Infinitary pp-definabillity over the real numbers with convex relations

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1. What are we talking about?
2. What is going on here?
3. Why is this interesting?

Infinitary pp-definabillity

Definition

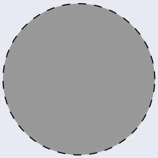
σ-structure X. The infinitary primitive positive closure (ipp-closure) is the smallest set of relations containing σ, closed under

- existentially quantification
- adding unused variables and
- finite **or infinite** conjunctions with finitely many free variables.

Examples

Take the open unit circle $B \subseteq \mathbb{R}^2$.

Definition (Circle)



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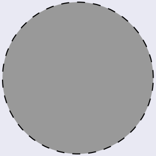

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$$\{x \mid \exists y\colon (x, y) \in B\} \in \mathbb{R}$$

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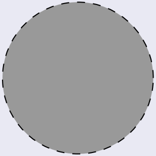

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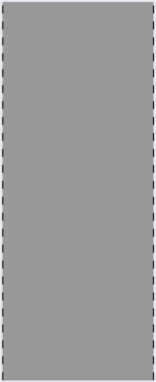
Definition (Infinite Rectangle)

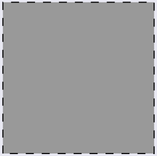


$$\{(x, y) \mid \exists z \colon (x, z) \in B\} \in \mathbb{R}^2$$

Examples

Take the open unit circle $B \subseteq \mathbb{R}^2$.

Definition (Square)



$$\{(x, y) \mid \exists a, b \colon (x, a) \in B \wedge (y, b) \in B\}$$

Question

Common relations

- addition $+$: $\{(x, y, z) \in \mathbb{R}^3 \mid x + y = z\}$.
- scalar multiplication $\cdot c \in \mathbb{R}$: $\{(x, y) \in \mathbb{R}^2 \mid cx = z\}$.
- constant $\{1\} \subseteq \mathbb{R}$.

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Answer

There are 6 possible closures.

They will be described later.

Immediate results

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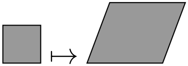
Question

Take the structure $(\mathbb{R}; +, \cdot c \mid c \in \mathbb{R}, 1, S)$ *where* S *is a convex set, what is the ipp-closure?*

- Every constant is definable.
- For an affine transformation



$$x \mapsto Ax + b$$

and a definable set Z, also its image and preimage

$$\{y \mid \exists x \colon x \in Z \wedge Ax + b = y\}$$
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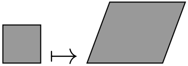

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- Every affine set is definable.

Immediate results

- We can reformulate the question:

Question

What is the smallest set containing S, constants, affine projections, affine preimages and arbitrary intersections?

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Immediate results

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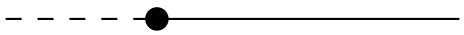
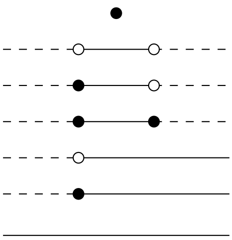
Question

What is the smallest set containing S, constants, affine projections, affine preimages and arbitrary intersections?

- Every ipp-definable set is convex.
- If S is affine, then every ipp-definable set is affine.
- If S is a ray () then every closed convex set is definable by the Hahn-Banach-Theorem.

First Classifications

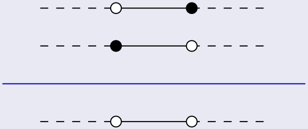
Table: Cassification of convex sets with dimension at most 1



empty set	$\emptyset$
single point	$\{0\}$
open interval	$(0, 1)$
halve open interval	$[0, 1)$
compact interval	$[0, 1]$
open ray	$(0, \infty)$
closed ray	$[0, \infty)$
real line	$(-\infty, \infty)$

First Classifications

Definition (Cutting edges)



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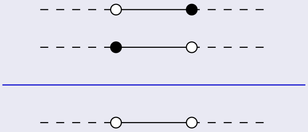

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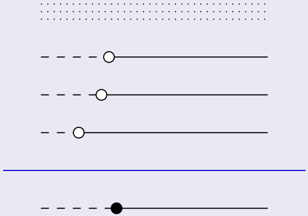

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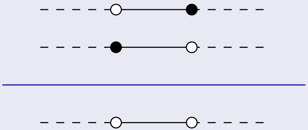

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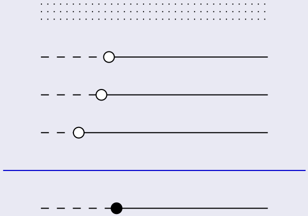
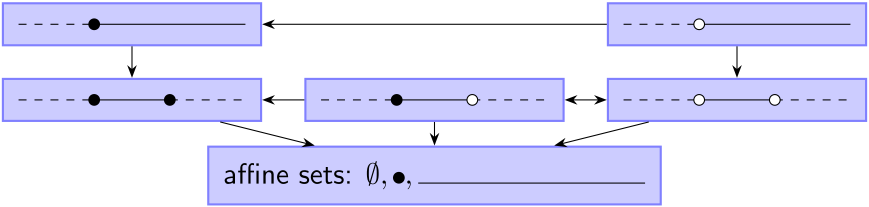
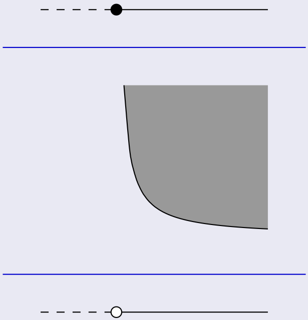




Higher Dimension

Definition (Obtaining non-closed sets)

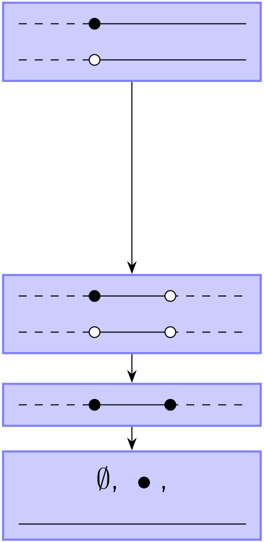


The closed set

$$\{(x, y) \mid x, y \geq 0 \wedge xy \geq 1\}$$

is ipp-definable from the closed ray. Its projection is the open ray.

Classification result



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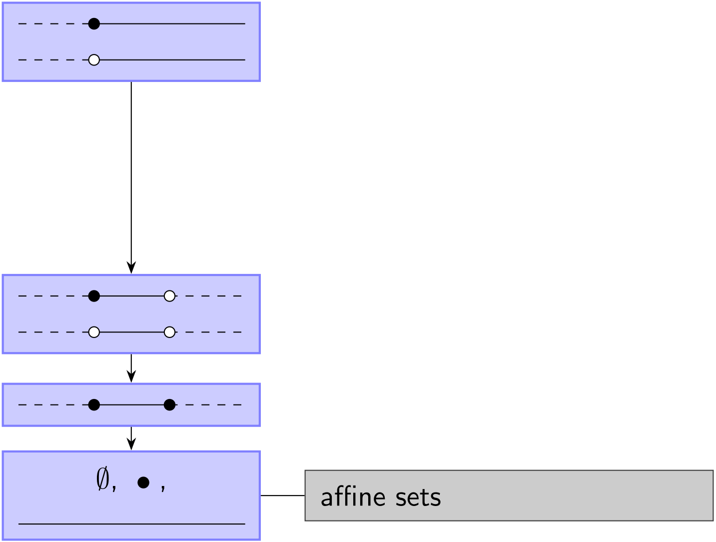

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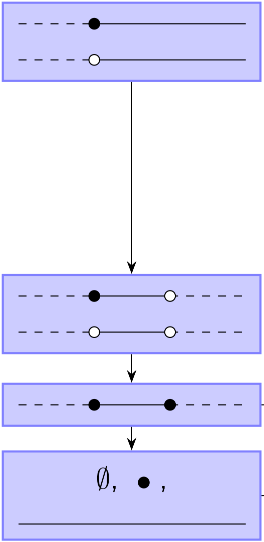
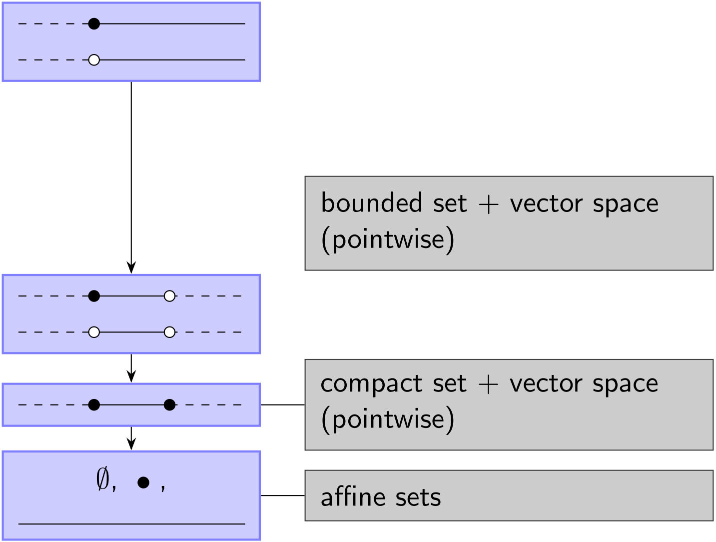


compact set + vector space (pointwise)

affine sets

Classification result



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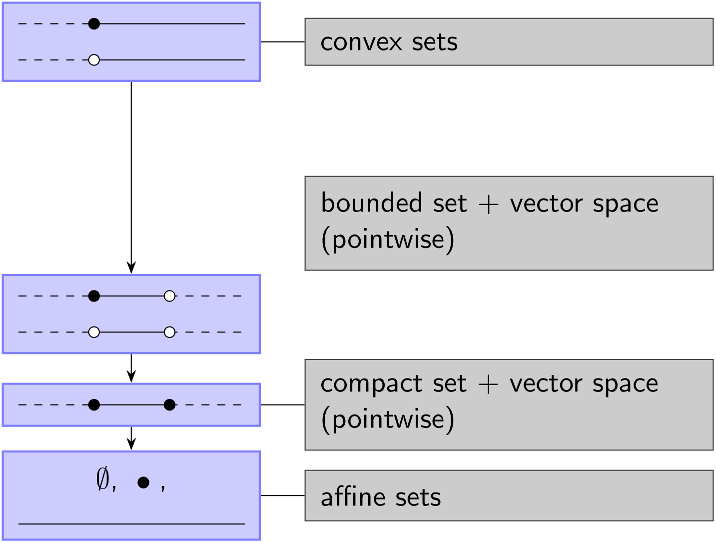

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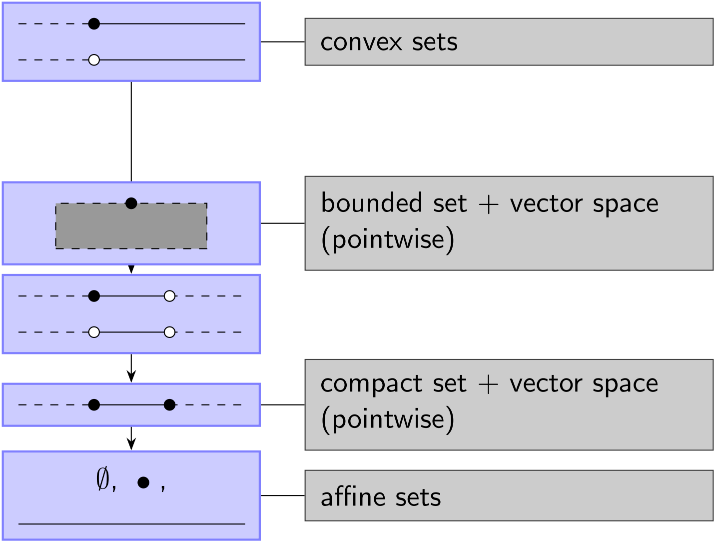

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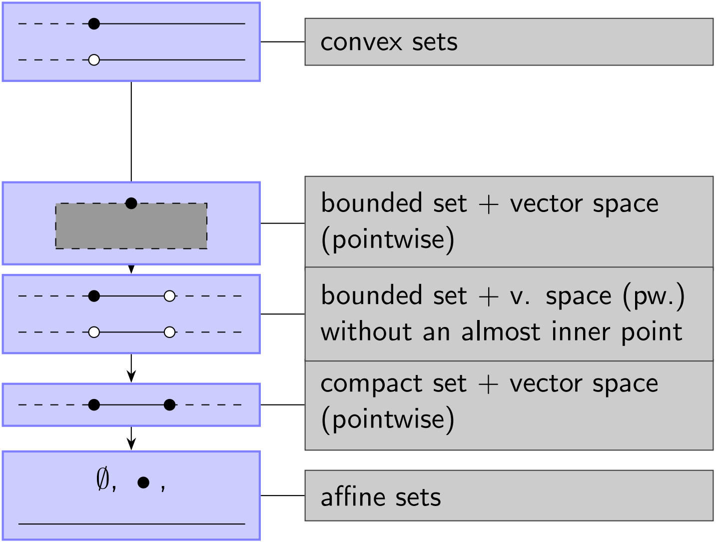

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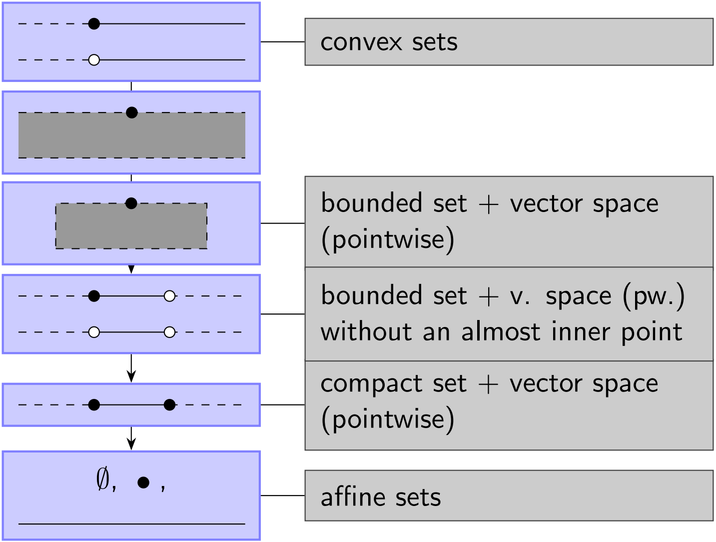

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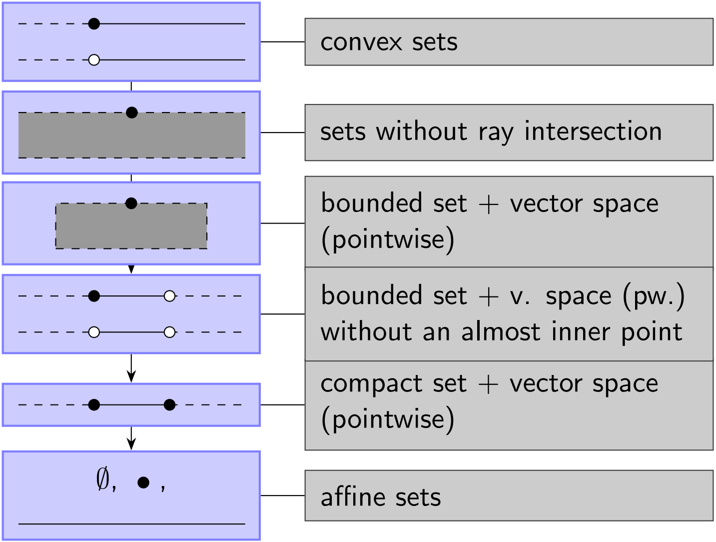
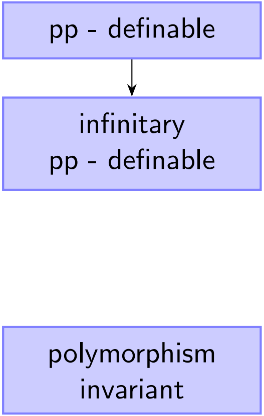


Applications

pp - definable

polymorphism
invariant

Applications



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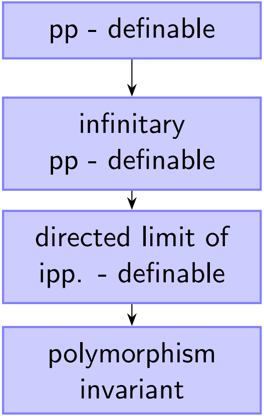

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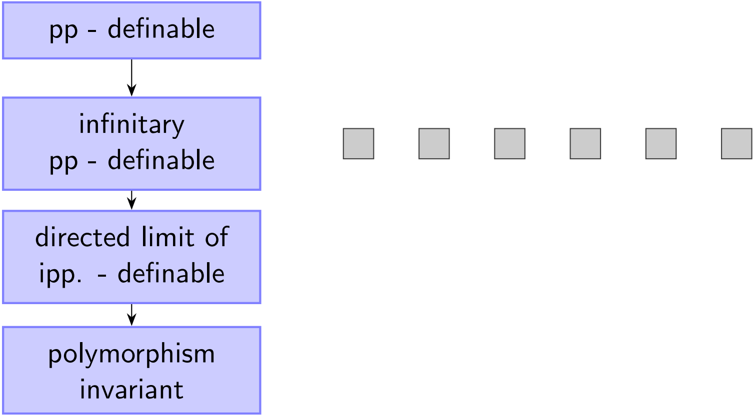

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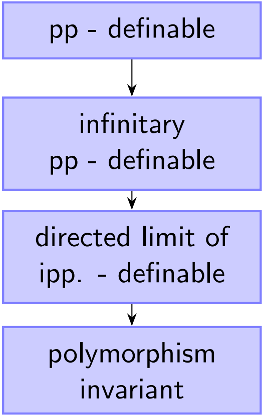

infinitely many equivalence classes

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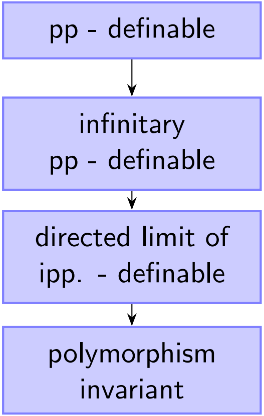

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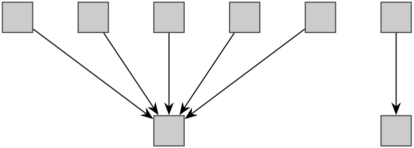

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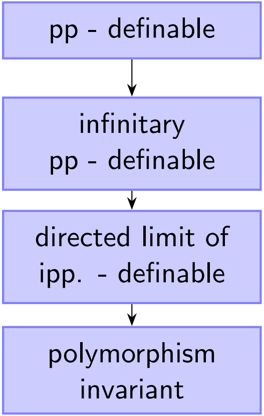

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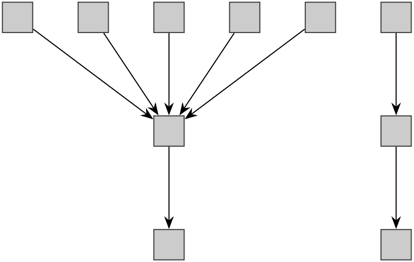

Applications

Definition

A map

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, ..., x_n) \mapsto \lambda_1 x_1 + ... + \lambda_n x_n$$

is called

- linear combination if $\lambda_1 + ... + \lambda_n = 1$ and
- convex combination if additionally $\lambda_1, ..., \lambda_n \geq 0$.

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Corollary

There is no locally closed clone between the clone of all linear combinations and the clone of all convex combinations on $\mathbb{R}$.

Thank you for your attention

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