Generic Structures for Monotone Monadic Second-Order Logic

Manuel Bodirsky

Institut für Algebra, TU Dresden
Joint work with Simon Knäuer and Sebastian Rudolph

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- New source of generic structures

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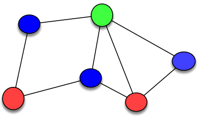

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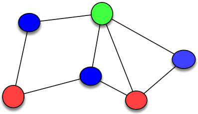

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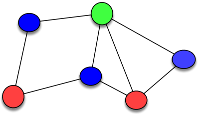

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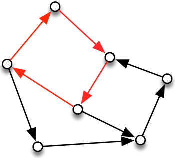

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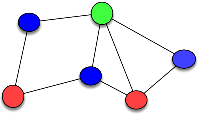

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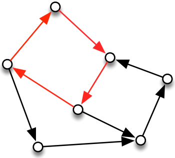

Observation. Both examples are monotone.

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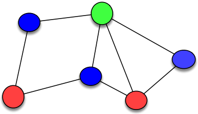

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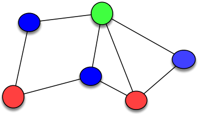

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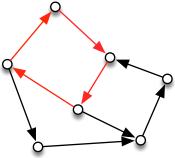

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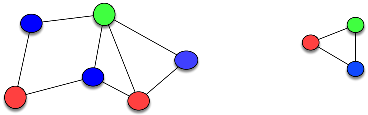

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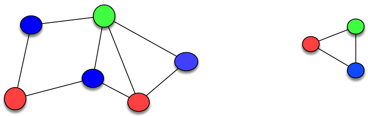

Further examples:

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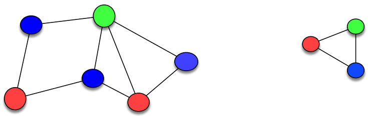

Further examples:

- CSP$(\mathbb{Q}; <)$: digraph acyclicity.
- CSP$(\mathbb{Q}; \text{Betw})$ where $\text{Betw} = \{(x, y, z) \mid x < y < z \lor z < y < x\}$: the Betweenness Problem, NP-complete.

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For every monotone MSO sentence Φ there exists a finite set of ω-categorical structures $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ such that

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every CSP in MSO equals $\mathrm{CSP}(\mathfrak{B})$ for an ω-categorical structure $\mathfrak{B}$.

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- Pol$(\mathfrak{A})$ as a topological clone (B.+Pinsker'15).

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Reformulation. If $\mathcal{C}$ is monotone, then there exists a finite set of finite structures $\mathcal{F}$ such that $\mathcal{C} = \text{Forb}(\mathcal{F})$ where

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Theorem (Cherlin+Shelah+Shi'99). Let $\mathcal{F}$ be a finite set of finite connected structures. Then there exists an ω-categorical model-complete structure $\mathfrak{B}$ such that $\mathfrak{A} \hookrightarrow \mathfrak{B}$ if and only if no structure in $\mathcal{F}$ has a homomorphism to $\mathfrak{A}$.

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Final step: If $\mathcal{F}$ contains structures that are not connected, find finitely many finite sets of finite connected structures $\mathcal{F}_1, \dots, \mathcal{F}_n$ such that

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Combining all this:

Corollary. There exists a finite set of ω-categorical structures $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ such that

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Then there exists an ω-categorical structure $\mathfrak{B}$ such that CSP$(\mathfrak{B}) = \mathcal{C}$
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Corollary. CSP$(\mathbb{Z}; \text{succ})$ is not expressible in MSO.

Remarks

- For every CSP $\mathcal{C}$ in MSO there exists a model complete ω-categorical structure $\mathfrak{B}$ such that Age$(\mathfrak{B}) = \mathcal{C}$, and $\mathfrak{B}$ is unique up to isomorphism.

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Consider the following MSO sentence Φ:

$$\forall X((\exists x.X(x)) \Rightarrow \exists x, y \in X \, \forall z \in X \big(\neg E(x,z) \vee \neg E(y,z)\big)$$

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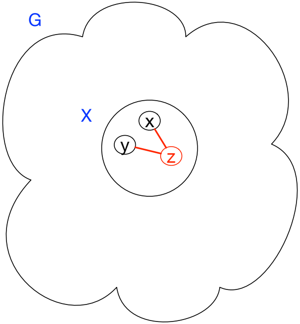

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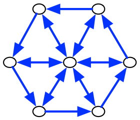
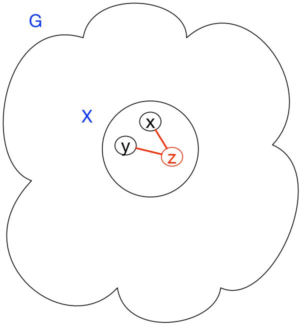
Expl. $W_i \not\models \Phi$ for every $i \geq 2$.



W_6



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GSO: additionally allow (unrestricted) second-order quantification.

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$$\exists L \, \forall x,y,z \Big(\mathrm{Betw}(x,y,z) \Rightarrow ((L(x,y) \wedge L(y,z)) \vee (L(z,y) \wedge L(y,x)) \wedge \underbrace{L \text{ is acyclic}}_{\text{in MSO}} \Big).$$

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Theorem (B.+Knäuer+Rudolph'21).

For every monotone GSO sentence Φ there exists a finite set of ω-categorical structures $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ such that

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