EPPA numbers of graphs

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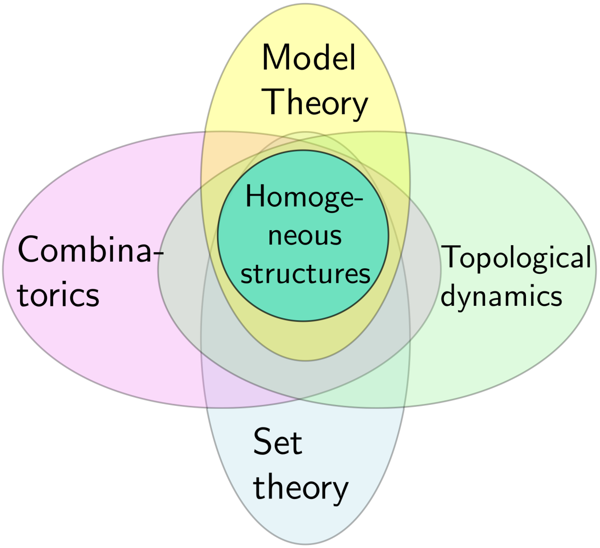
Model Theory
Combina-torics
Homoge-neous structures
Topological dynamics
Set theory

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Example

A graph **G** is vertex-transitive if every partial automorphism f with $|\mathrm{Dom}(f)| \leq 1$ extends to an automorphism of **G**.

Definition (EPPA, extension property for partial automorphisms)

Let **B** be a structure and let **A** be its **induced** substructure. **B** is an EPPA-witness for **A** if every partial automorphism of **A** extends to an automorphism of **B**.

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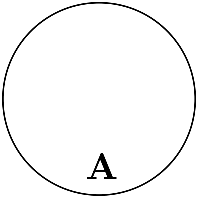

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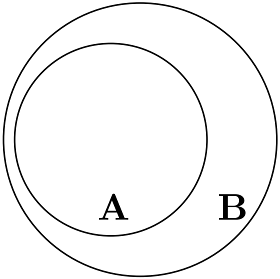

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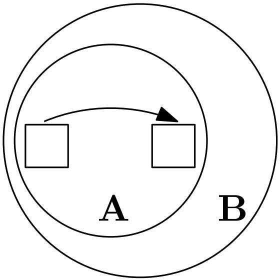

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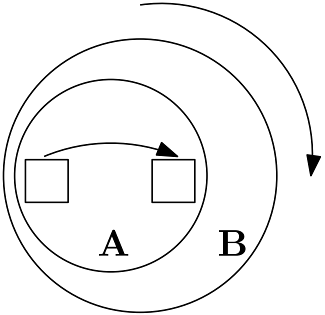

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Theorem (Hrushovski, 1992)

The class of all finite graphs has EPPA.

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Every class with EPPA has the amalgamation property and corresponds to a countable homogeneous structure.

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Theorem (Kechris–Rosendal, 2007)

If $\mathbf{M}$ *homogeneous then* $\mathrm{Aut}(\mathbf{M}) = \overline{\bigcup_i G_i}$ *with compact* $G_1 \leq G_2 \leq \cdots \mathrm{Aut}(\mathbf{M})$ *if and only if* $\mathrm{Age}(\mathbf{M})$ *has EPPA.*

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EPPA is known to hold for graphs, K_n-free graphs, hypergraphs, metric spaces, free amalgamation classes, two-graphs, finite groups, . . .

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Problem (Hrushovski, 1992)

Improve the bounds.

Theorem (Herwig–Lascar, 2000)

For every $\mathbf{G}$ *with* n *vertices and maximum degree* Δ *we have that* $\mathrm{eppa}(\mathbf{G}) \leq \binom{\Delta n}{\Delta} \in n^{\mathcal{O}(n)}$.
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Theorem (Evans–Hubička–K–Nešetřil, 2021)

For every graph $\mathbf{G}$ *it holds that* $\mathrm{eppa}(\mathbf{G}) \leq n2^{n-1}$.

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Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is k-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{k}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
6. Every permutation of E induces an automorphism of $\mathbf{H}$. □

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For non-k-regular graphs, add "half-edges" to make them regular.

An upper bound [Evans, Hubička, K, Nešetřil 2021]

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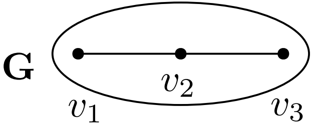

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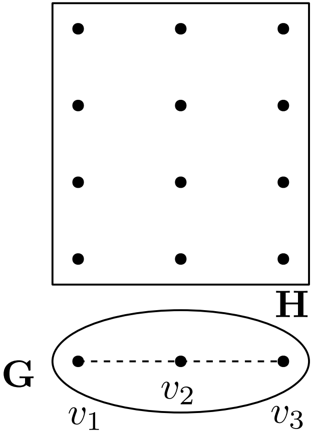

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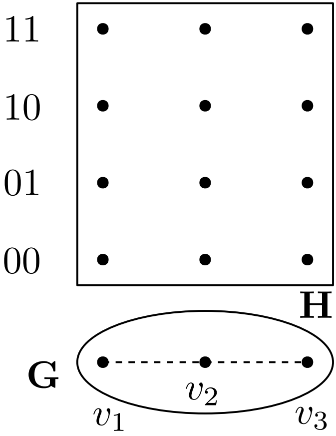

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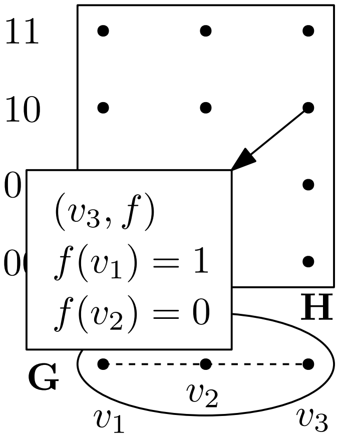

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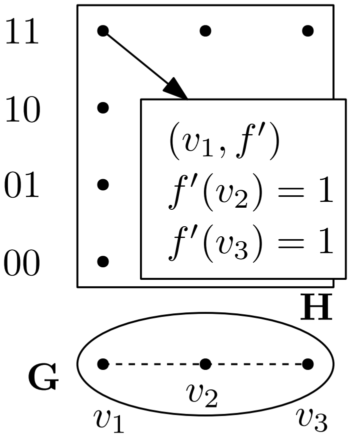

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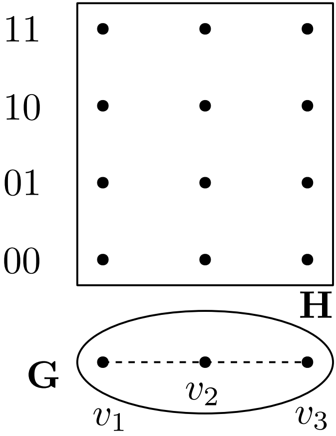

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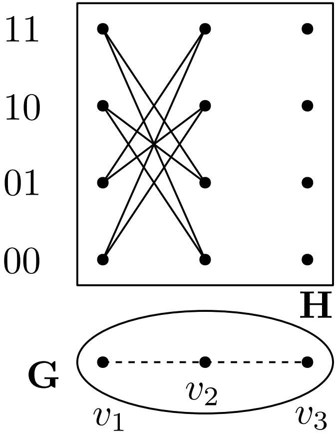

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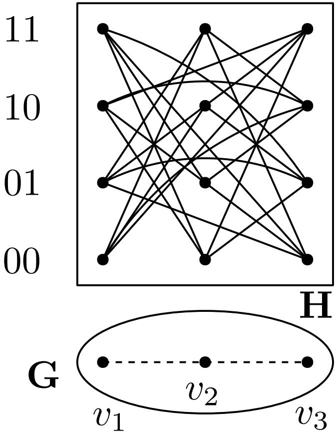

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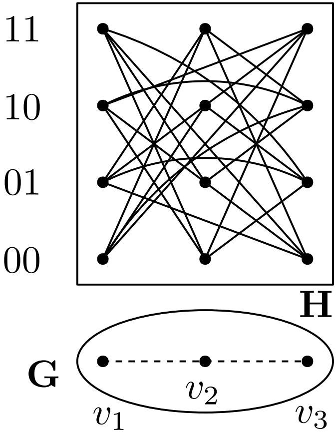

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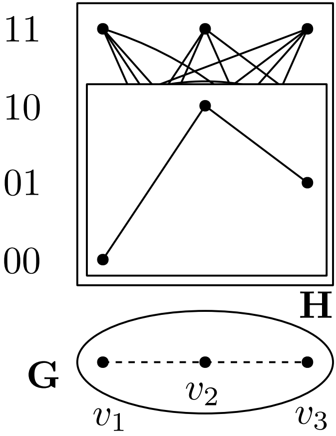

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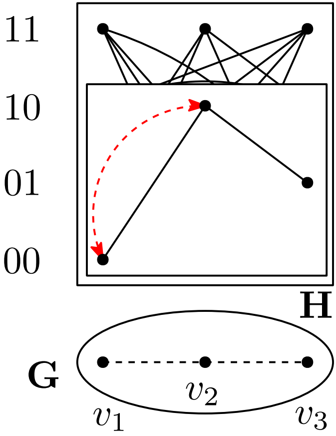

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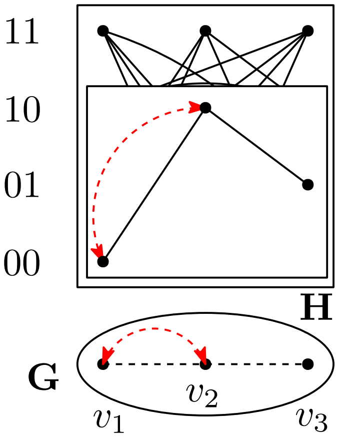

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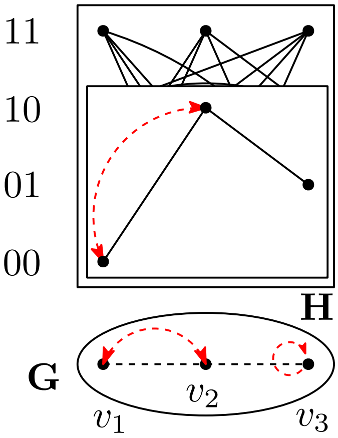

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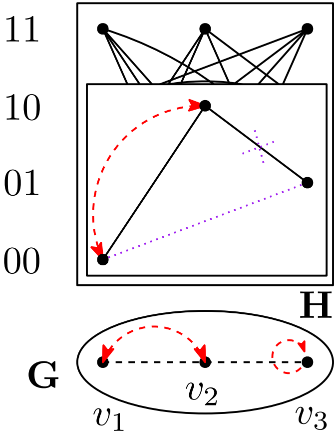

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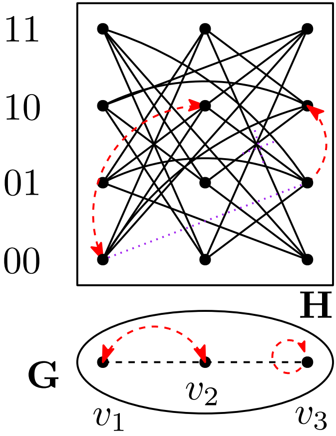

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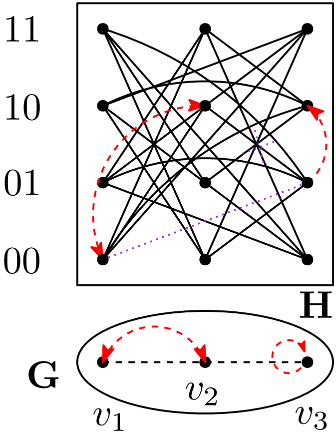

For $u, v \in G$, we define a flip $F_{u,v}((w, f)) = (w, f')$, where

$$f'(x) = \begin{cases} 1 - f(x) & \text{if } \{w, x\} = \{u, v\} \\ f(x) & \text{otherwise.} \end{cases}$$

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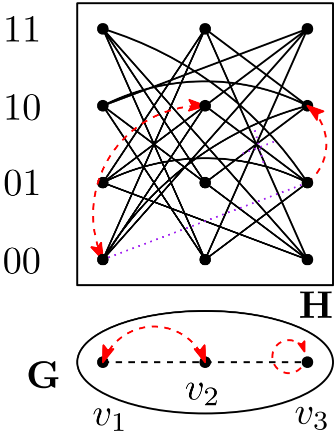

Remark

This can be straightforwardly generalised to arbitrary relational structures and less straightforwardly one can also add unary functions.

Conclusion

- We have $\sqrt{2}^n \leq \max\{\mathrm{eppa}(\mathbf{G}) : |G| = n\} \leq n2^{n-1}$. Can this exponential gap be closed?
- The half-graph is an important example in e.g. model theory. Is there something going on here? (Cf. the Malliaris–Shelah Regularity Lemma for edge-stable graphs.)
- For graphs with maximum degree Δ we have $\mathrm{eppa}(\mathbf{G}) \in \mathcal{O}(n^\Delta)$, but no lower bound. Can a lower bound be proved? At least for cycles?
- What are the EPPA numbers of $G(n, 1/2)$? Can one prove at least a non-linear lower bound?

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- What are the EPPA numbers of $G(n, 1/2)$? I STILL HAVE NO IDEA AND IT MAKES ME SAD. Can one prove at least a non-linear lower bound? I think so..?

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Cycles have quadratic EPPA numbers.

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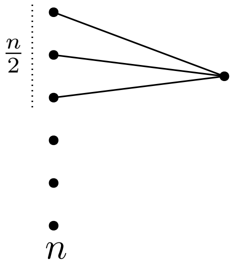

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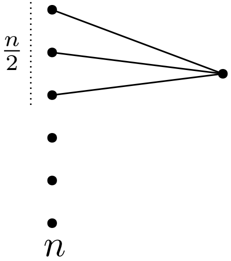

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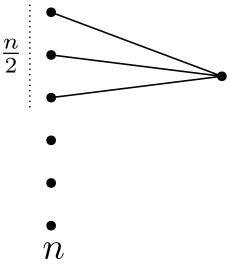

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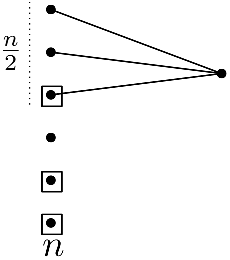

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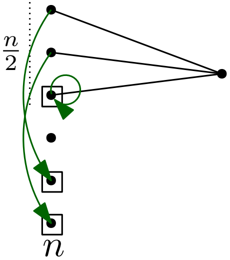

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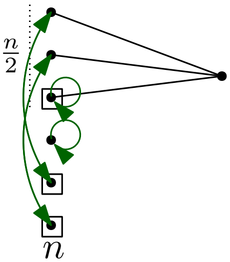

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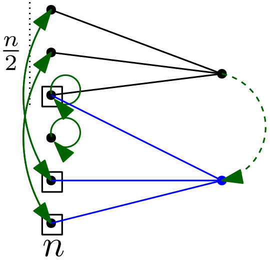

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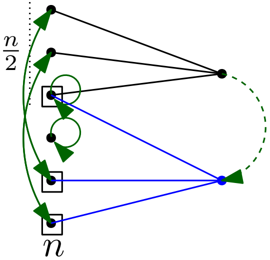

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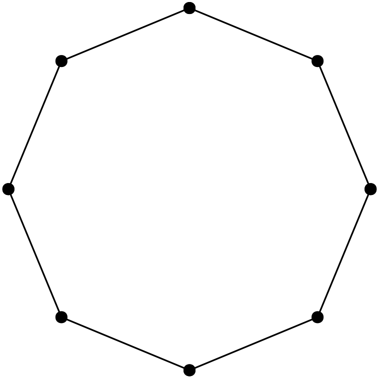

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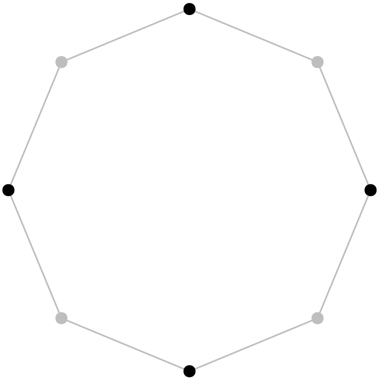

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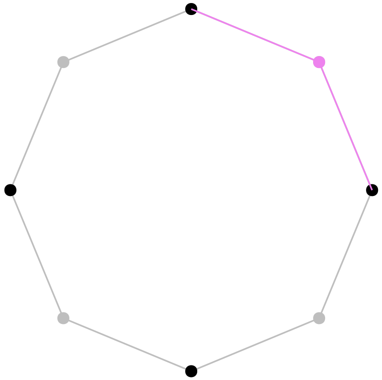

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Prove (or disprove) that $\mathrm{eppa}(G(n, 1/2))$ *is superexponential.*

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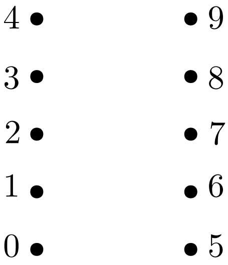

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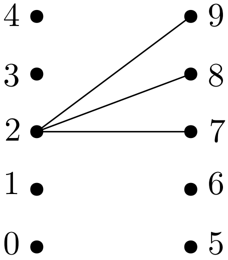

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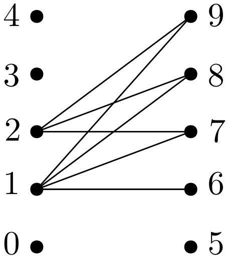

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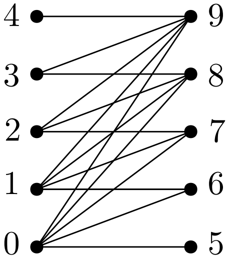

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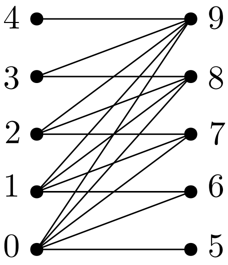

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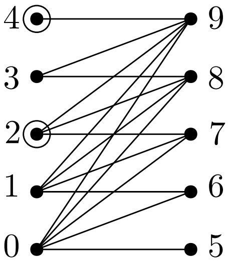

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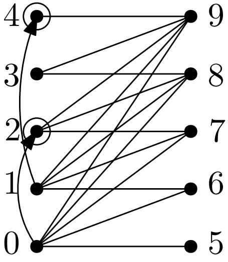

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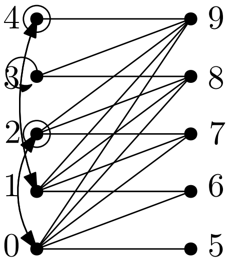

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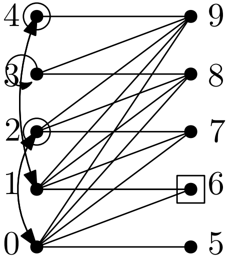

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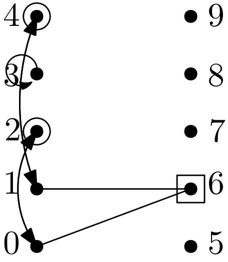

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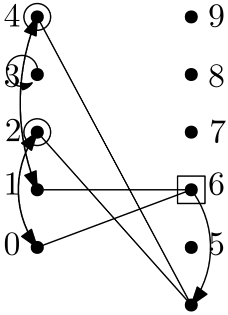

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