SYMMETRIES OF STRUCTURES THAT FAIL TO INTERPRET SOMETHING FINITE

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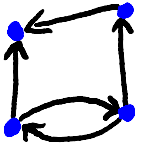
ERC SYNERGY GRANT POCOCOP (GA 101071674)
FWF I5948

$\mathbb{A}$ ... DIGRAPH / RELATIONAL STRUCTURE (POSSIBLY INFINITE)



CSP($\mathbb{A}$)

GIVEN: • VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \to x_2$, $x_2 \to x_3$, $x_3 \to x_1$, ...

QUESTION: $\exists$ SOLUTION IN $\mathbb{A}$?

EXAMPLES

- $\mathbb{A}$ =  3-COLORING
- $\mathbb{A}$ =  2-COLORING

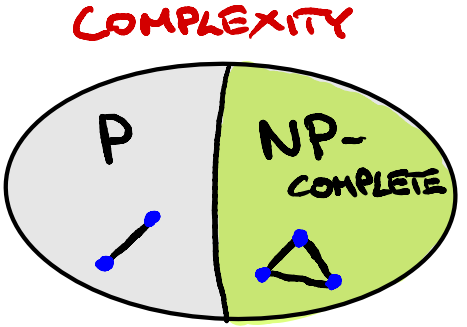
} COMBINATORIAL PROBLEMS

- $\mathbb{A} = (\mathbb{Q}, <)$ =  DIGRAPH ACYCLICITY
- $\mathbb{A} = (\mathbb{Q}, \text{BETWEEN}(x,y,z))$ =  BETWEENNESS PROBLEM

} LOGIC PROBLEMS
EG. HMSNP MODEL CHECKING

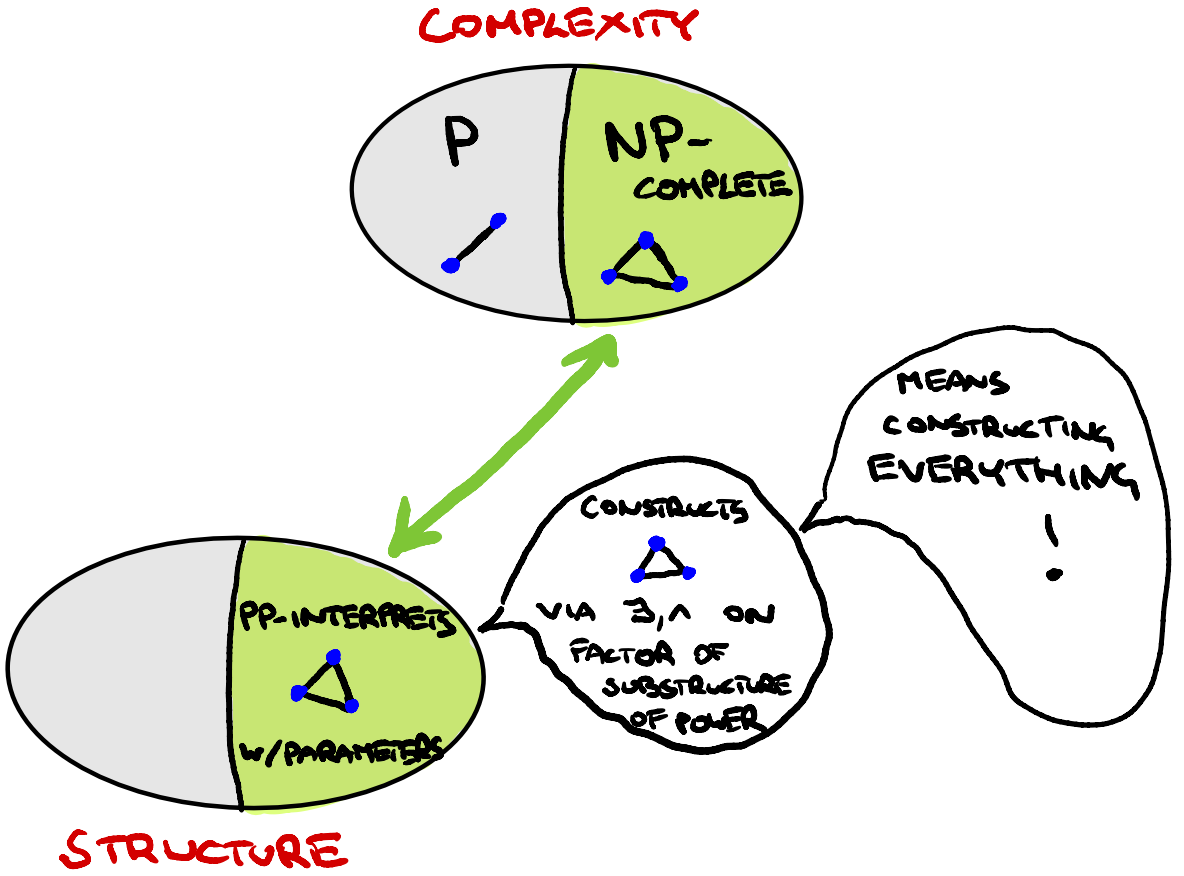
THEOREM (BULATOV, ZHUK '17 - CONJECTURE OF FEDER+VARDI 90s)

$\mathbb{A}$ FINITE



THEOREM

A FINITE



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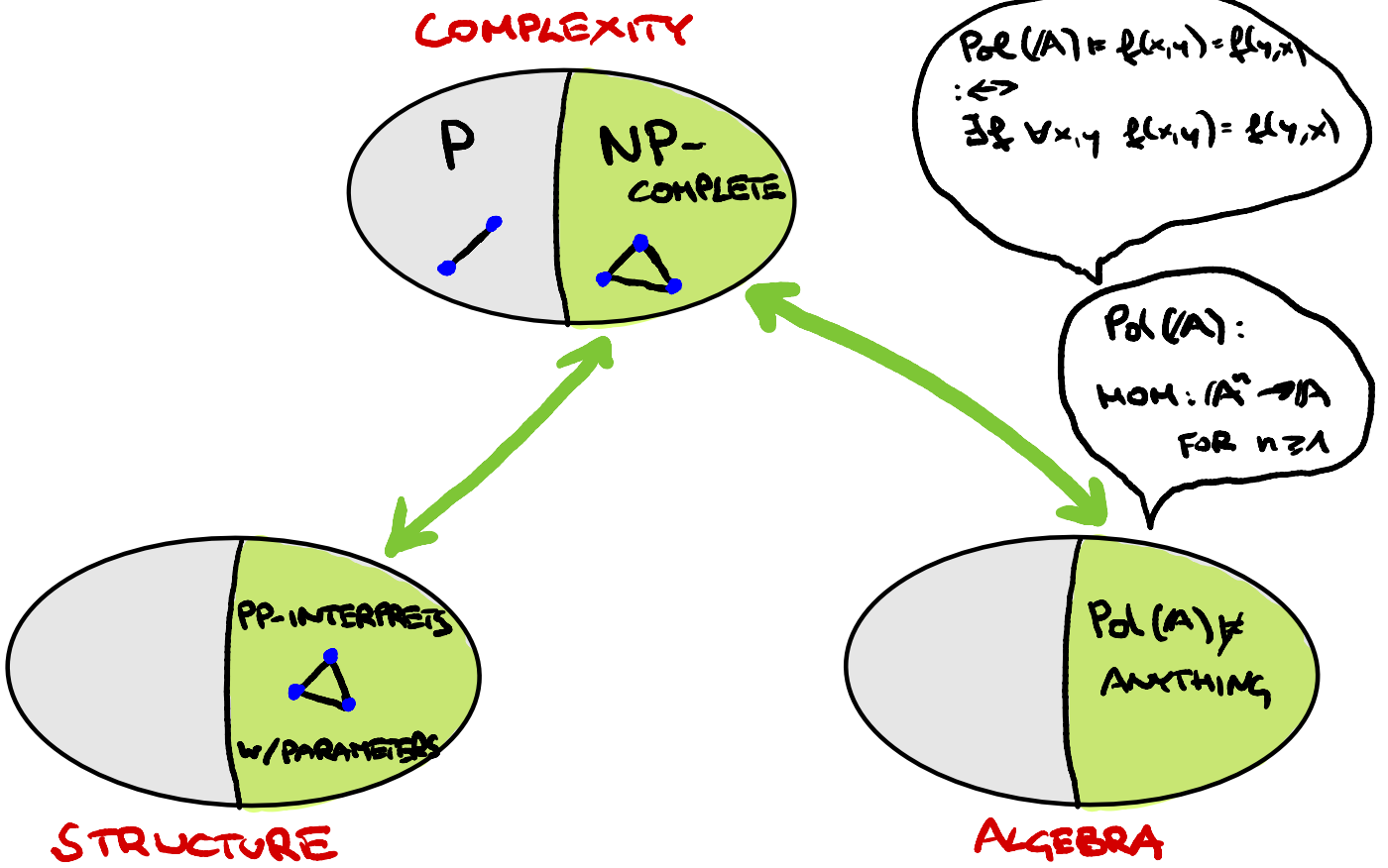

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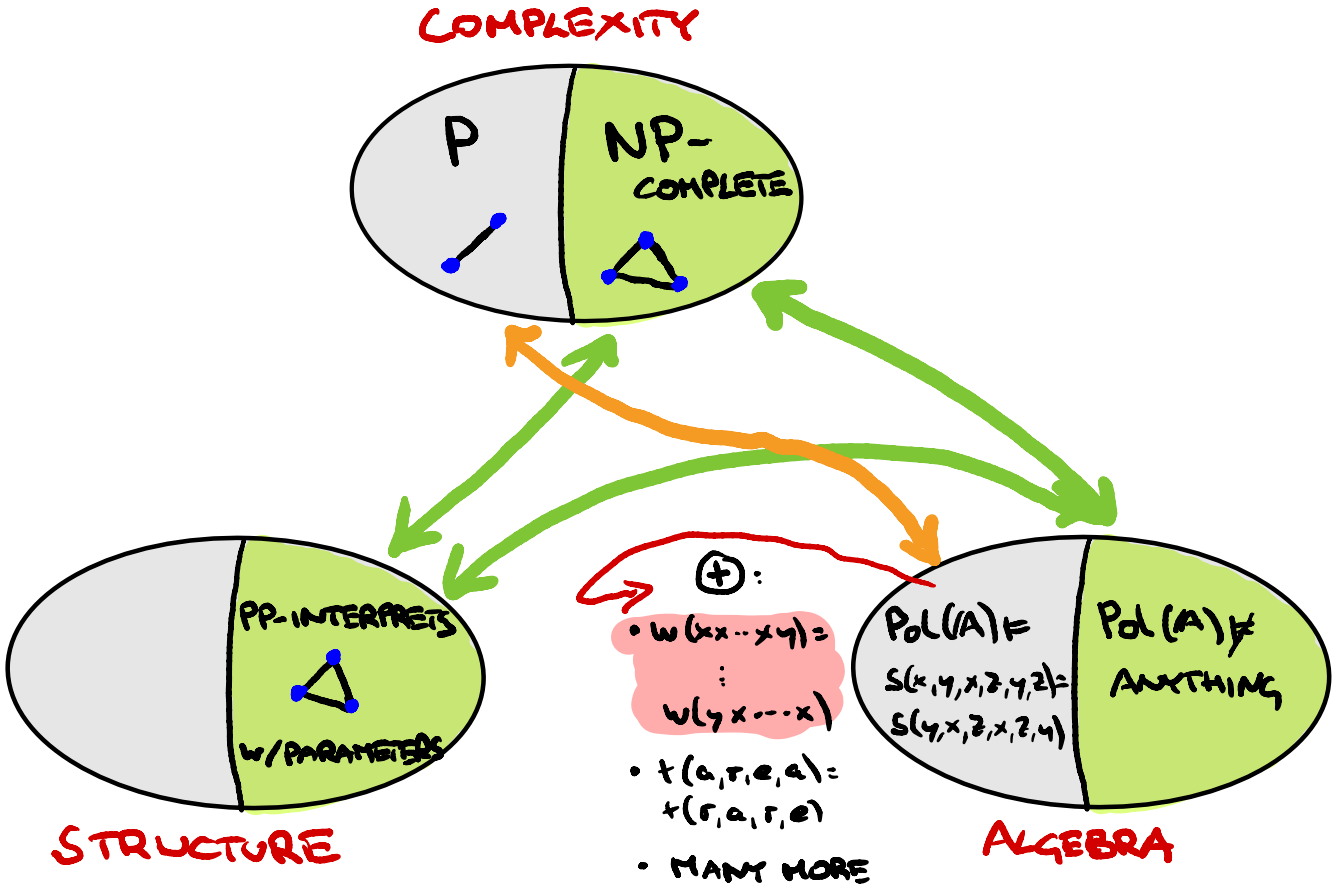

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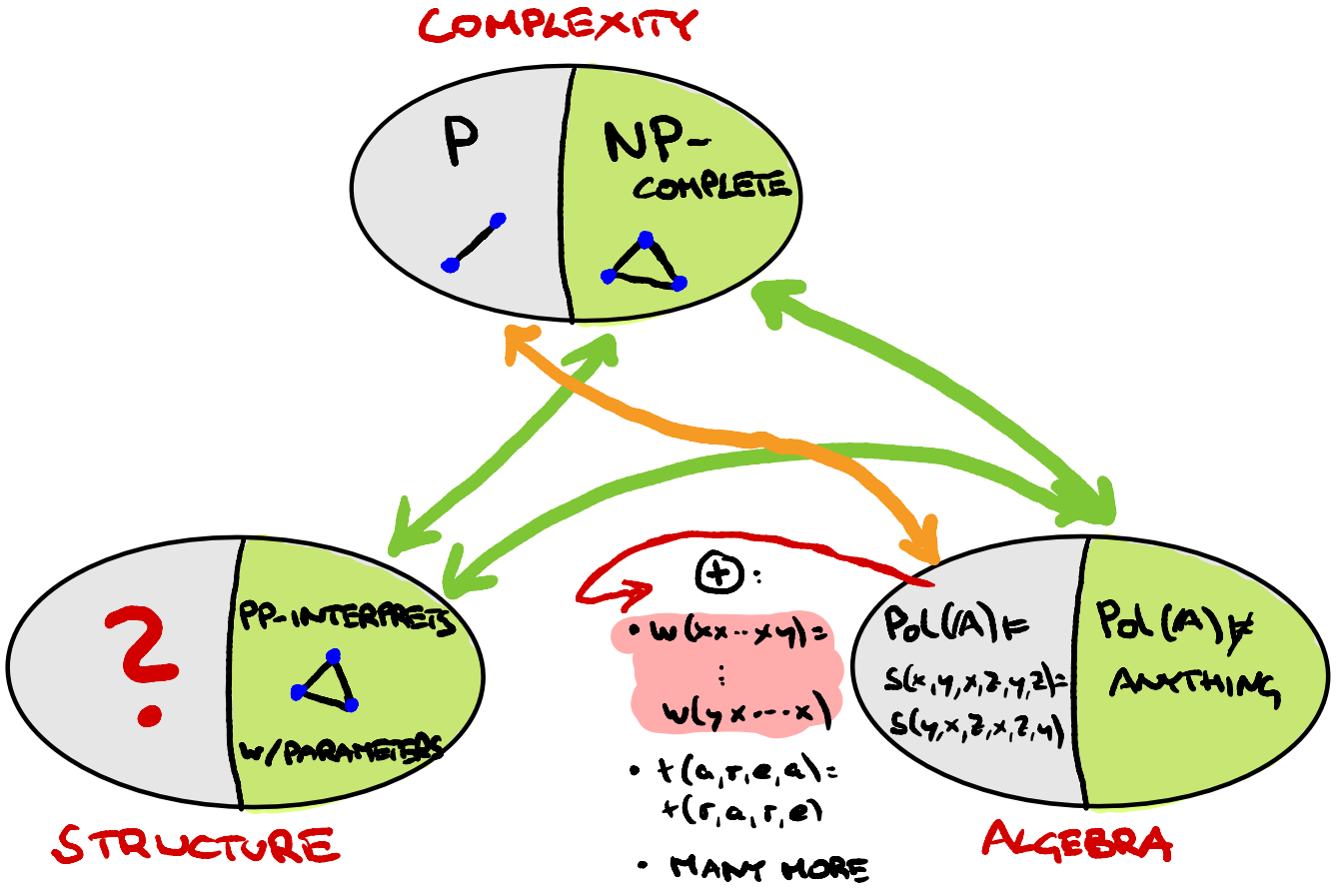

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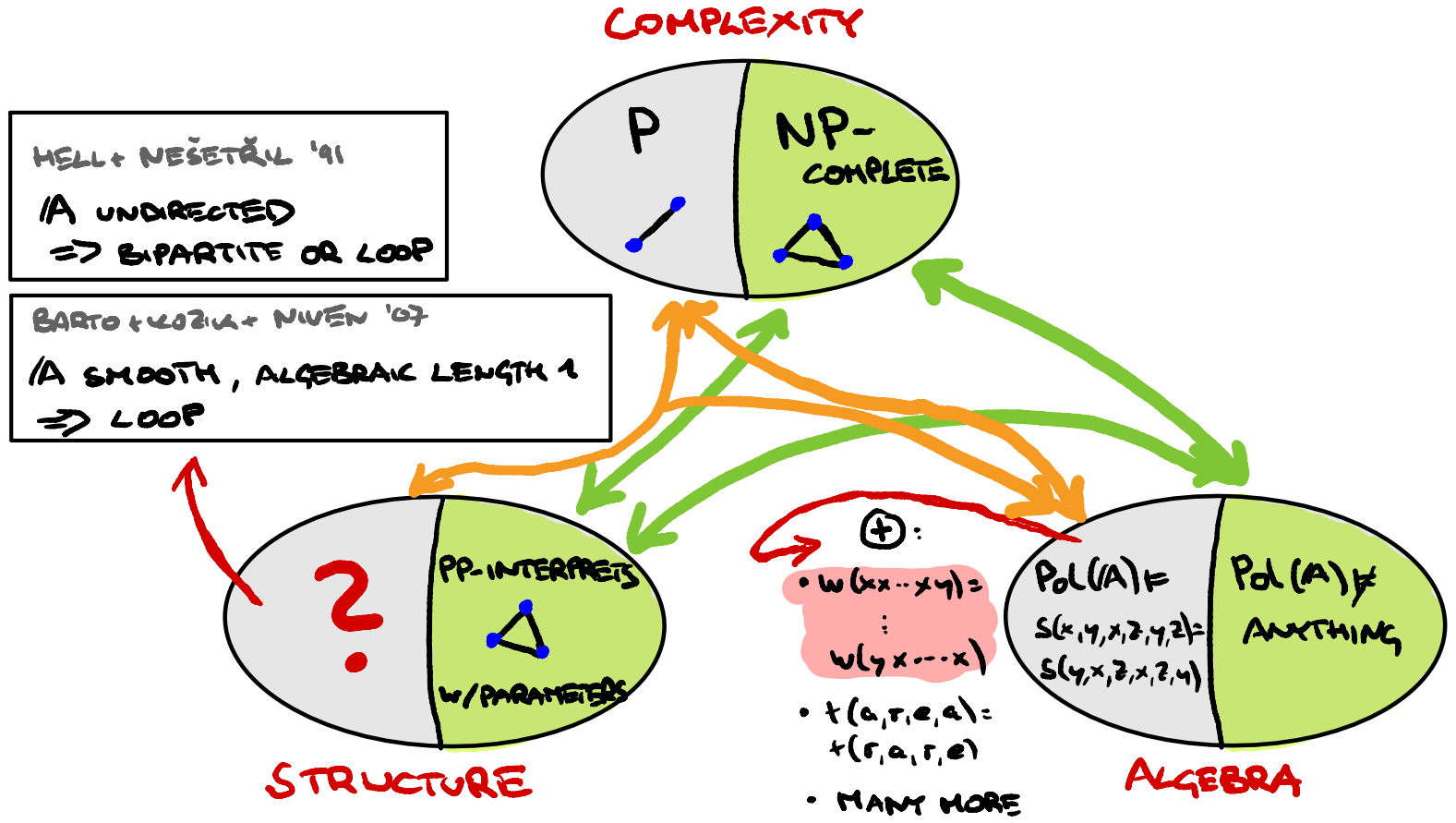


THEOREM

$\mathbb{A}$ FINITE

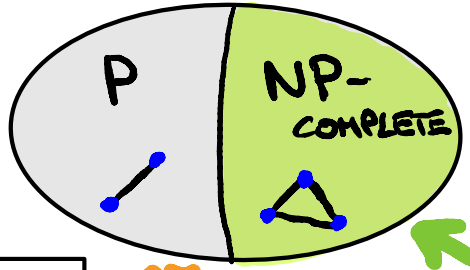
WANTED

③ INFINITE $\mathbb{A}$

① STRUCTURE PROOFS

② STRUCTURE RESULTS
E.G. NO PARAMETERS

COMPLEXITY

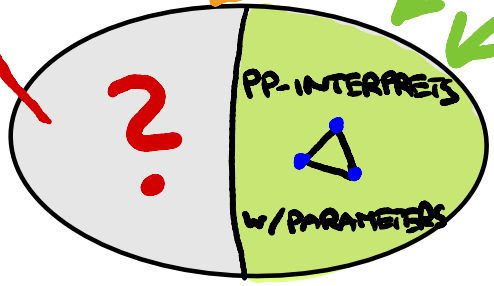


HELL + NEŠETŘIL '91

$\mathbb{A}$ UNDIRECTED
⇒ BIPARTITE OR LOOP

BARTO + KOZIK + NIVEN '07

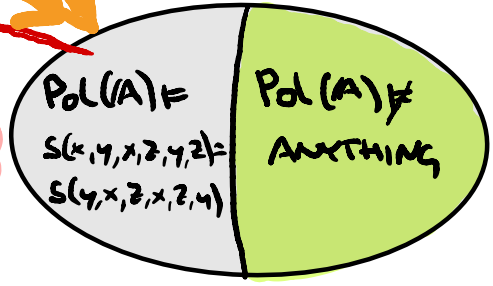
$\mathbb{A}$ SMOOTH, ALGEBRAIC LENGTH 1
⇒ LOOP



STRUCTURE

⊕:

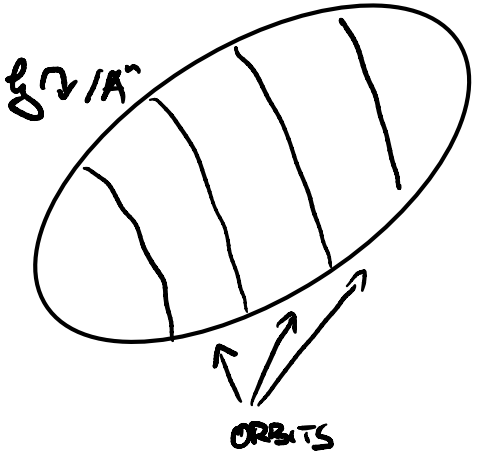
- $w(xx\cdots xy) =$ $\vdots$ $w(yx\cdots x)$
- $t(a,r,e,a) = t(r,a,r,e)$
- MANY MORE



ALGEBRA

$\mathbb{A}$ INFINITE BUT $\exists G \leq \mathrm{Aut}(\mathbb{A})$:

- $\forall n$ $\mathbb{A}^n/G$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION



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E.G. $(\mathbb{Q}, <)$

$\mathbb{A}$ FO-DEFINABLE IN FINITELY BOUNDED HOMOGENEOUS STRUCTURE

CONJECTURE (BODIRSKY + P. '11)

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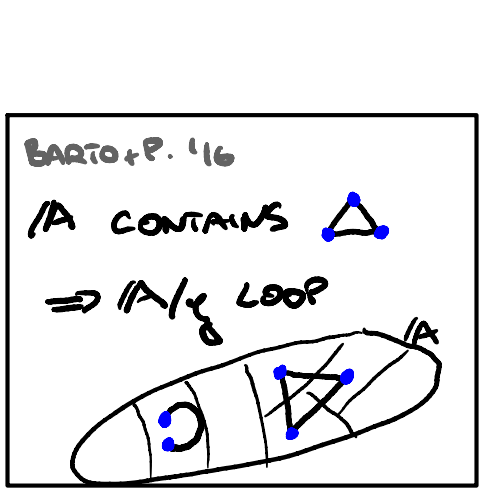

STRUCTURE

THEOREM "BETTER BARTO-KOZIK-NIVEN"

$G \leq \mathrm{Aut}(\mathbb{A})$, $\mathbb{A}$ FINITE

$\mathbb{A}$ SMOOTH, CONNECTED, ALGEBRAIC LENGTH 1

$\neg$ PP-INTERPRETS  W/ G-ORBITS

$\Rightarrow$ LOOP 

THEOREM „INFINITE HELL-NEŠETŘIL"

$G \leq \mathrm{Aut}(\mathbb{A})$, $\mathbb{A}/G$ FINITE

$\mathbb{A}$ SMOOTH, $\mathbb{A}/G$ SYMM., NON-BIPARTITE

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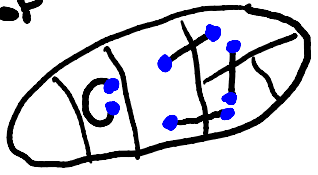
$\Rightarrow \mathbb{A}/G$ LOOP



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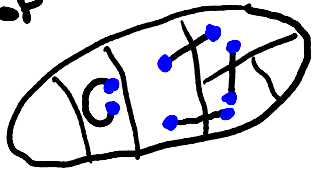

ALGEBRA

COROLLARY

$\mathbb{A}$ FINITE

$\not\to$ PP-INTERPRETS  (NO PARAMETERS)

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A}) \in$ $S(\alpha_1 x, \dots, \alpha_m x, x, y, x z, y z) =$
$S(y \ \dots \ y \ y \ x z \ x z)$

COROLLARY

$\mathbb{A}/G$ FINITE

$\not\to \mathbb{A}$ PP-INTERPRETS  W/ PARAMETERS

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A}) \in$ MANY THINGS

STRUCTURE

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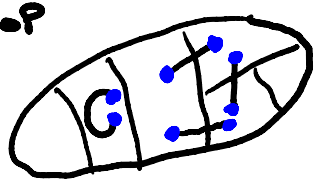

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COUNTEREXAMPLE

$\alpha_1 w(x \cdots x y) = \cdots = \alpha_n w(y x \cdots x)$

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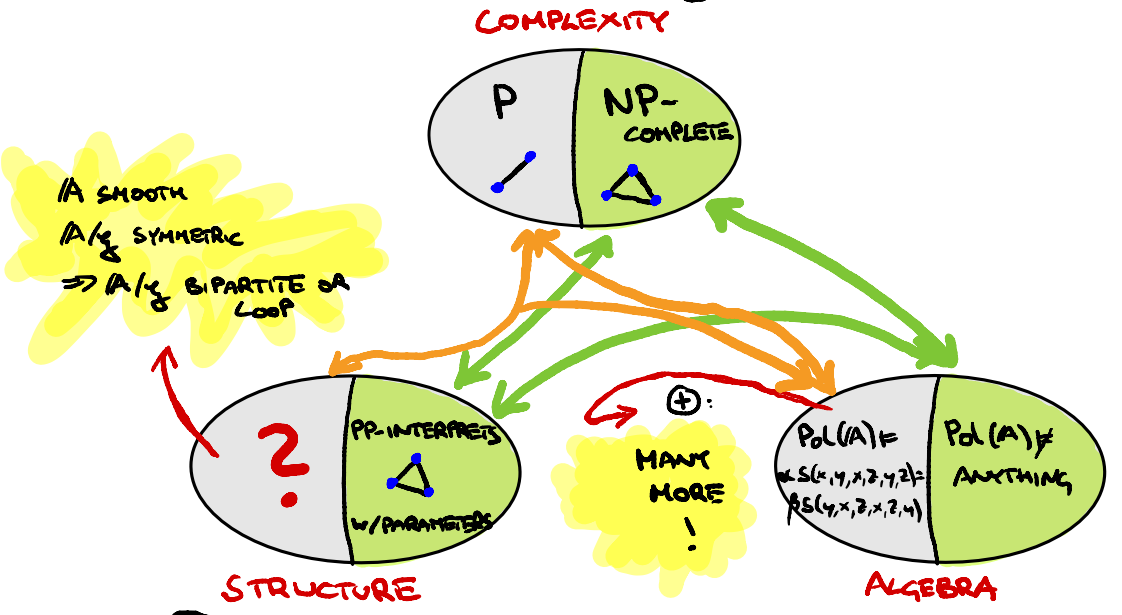

HO!

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THERE WERE A COUPLE OF LIES IN THIS TALK.