

SYMMETRIES OF STRUCTURES THAT FAIL TO INTERPRET SOMETHING FINITE

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LICS 2023

TU WIEN

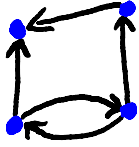


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ERC SYNERGY GRANT POCOCOP (CA 101071674)
FWF I5948

HO!

$\mathcal{A} \dots$ DIGRAPH / RELATIONAL STRUCTURE (POSSIBLY INFINITE)




CSP(\mathcal{A})

GIVEN: • VARIABLES x_1, \dots, x_n

• CONSTRAINTS $x_1 \rightarrow x_2, x_2 \rightarrow x_3, x_3 \rightarrow x_1, \dots$


QUESTION: \exists SOLUTION IN \mathcal{A} ?

EXAMPLES

• $\mathcal{A} =$  3-COLORING

• $\mathcal{A} =$  2-COLORING

• $\mathcal{A} = (\mathbb{Q}, <) =$  DIGRAPH ACYCLICITY

• $\mathcal{A} = (\mathbb{Q}, \text{BETWEEN}(x, y, z)) =$  BETWEENNESS PROBLEM

COMBINATORIAL PROBLEMS

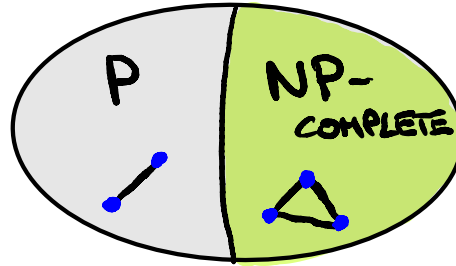
LOGIC PROBLEMS

EG. HMSNP MODEL CHECKING

THEOREM (BULATOV, ZHUK '17 - CONJECTURE OF FEDER+VARDI 90s)

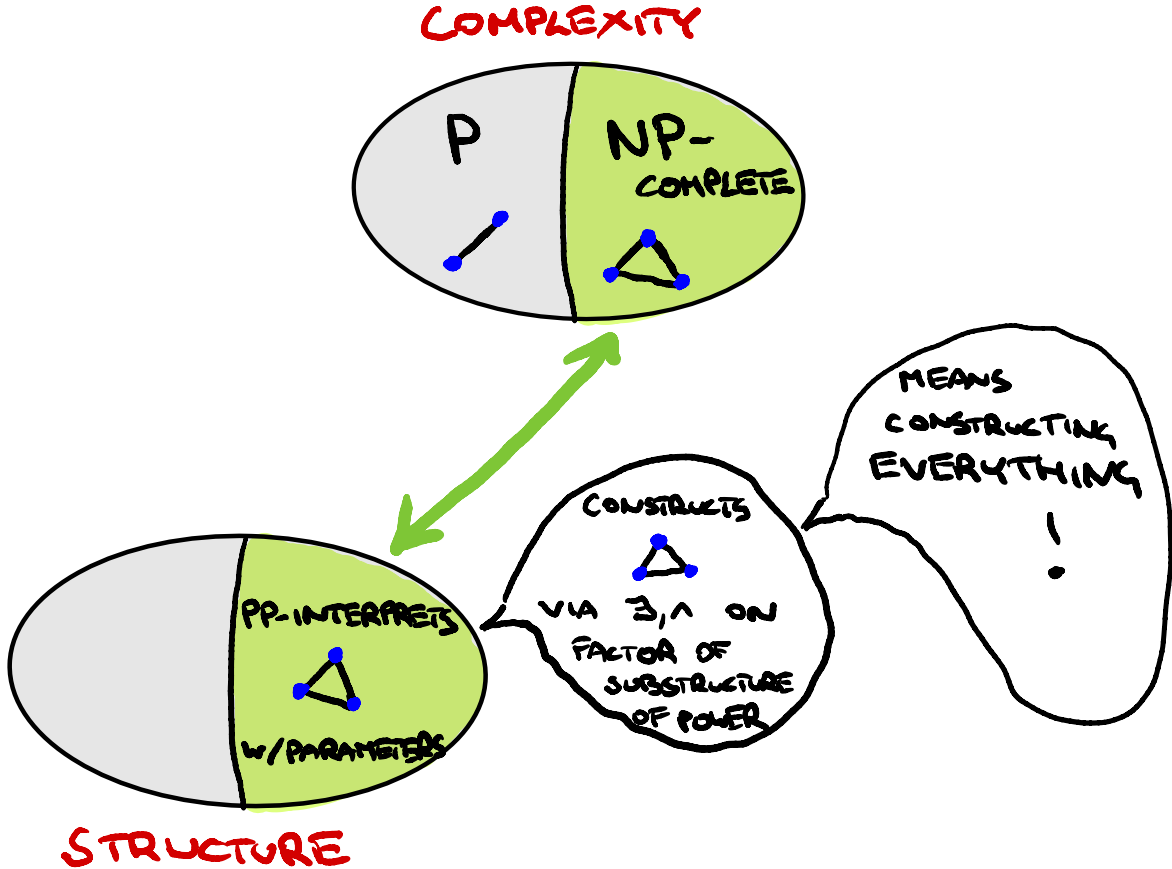
A FINITE

COMPLEXITY



THEOREM

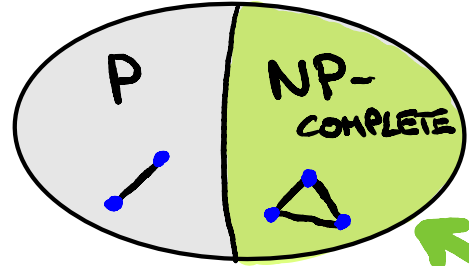
A FINITE



THEOREM

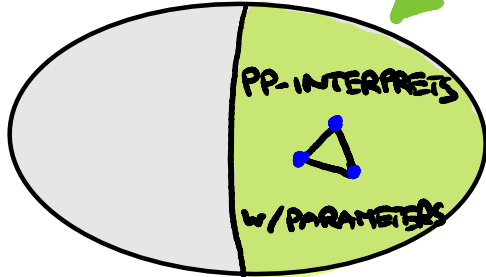
A FINITE

COMPLEXITY

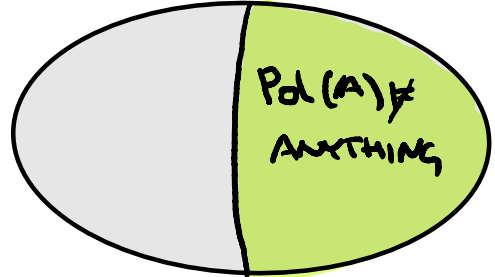


$Pol(A) = \{f(x,y) = f(y,x)\}$
: \Leftrightarrow
 $\exists f \forall x,y f(x,y) = f(y,x)$

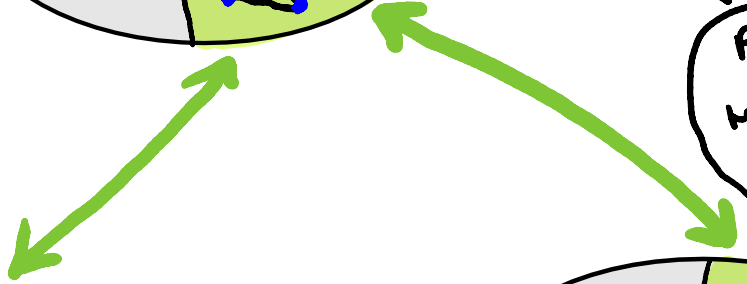
$Pol(A)$:
MON: $(A^n \rightarrow A)$
FOR $n \geq 1$



STRUCTURE

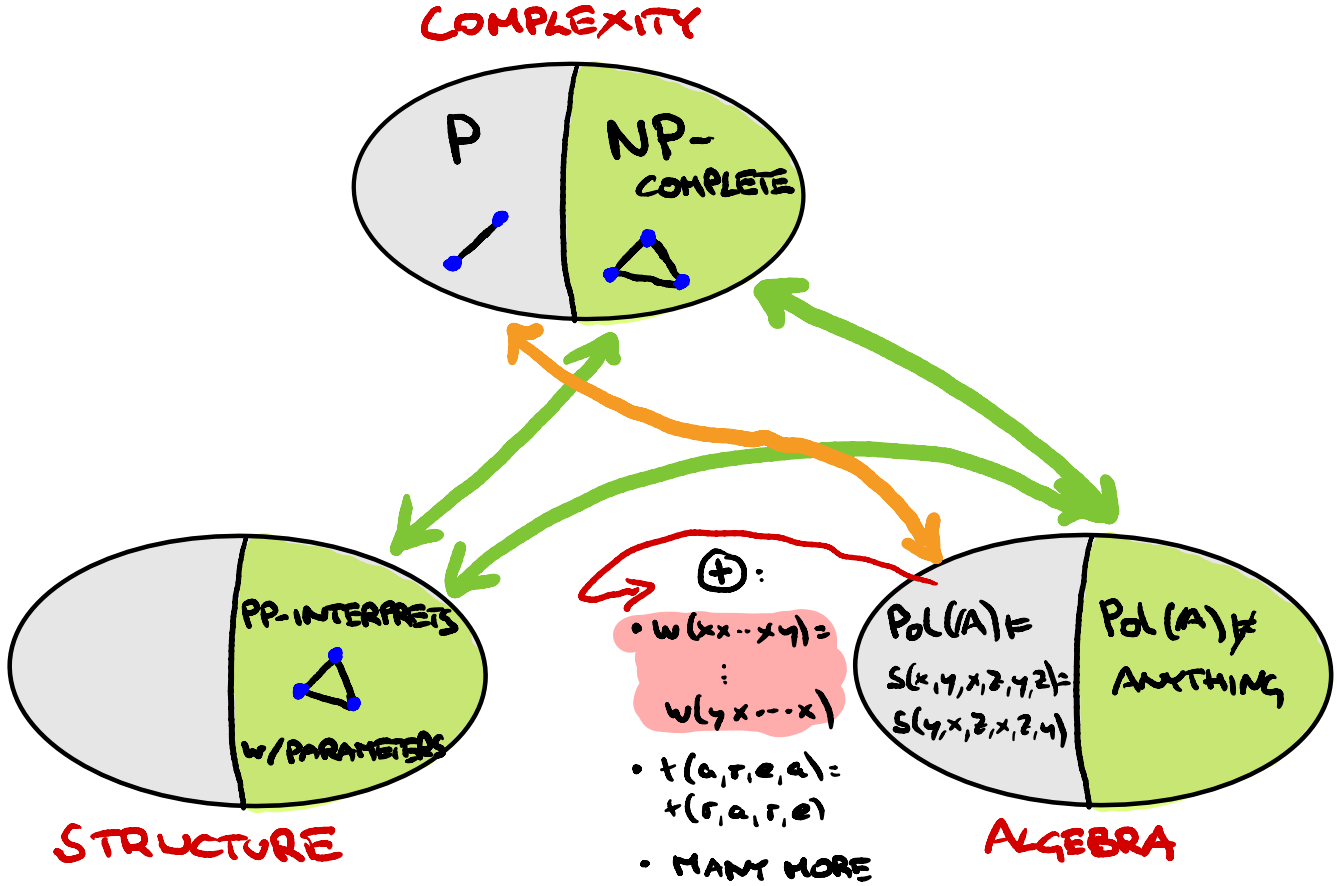


ALGEBRA



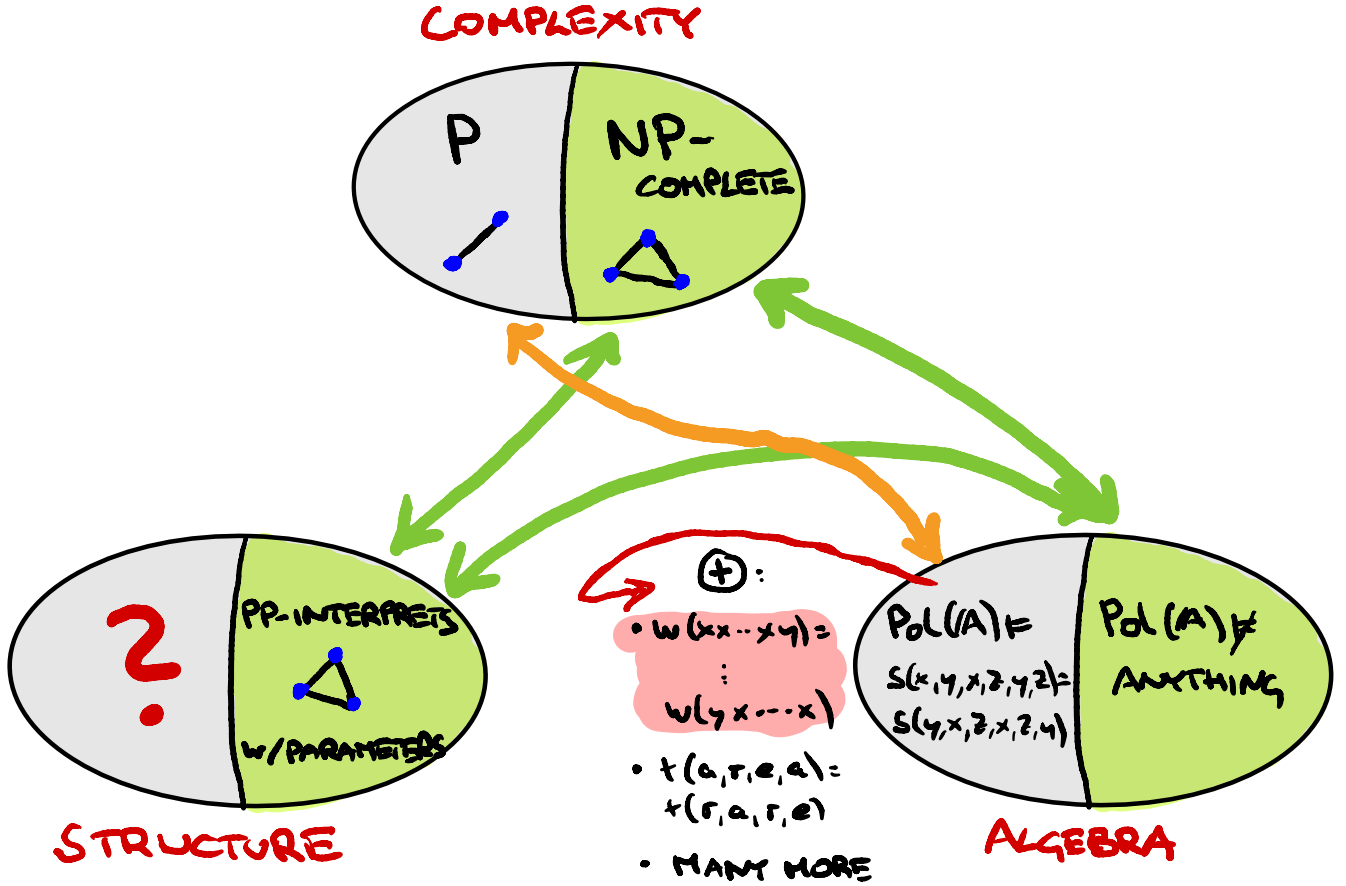
THEOREM

A FINITE



THEOREM

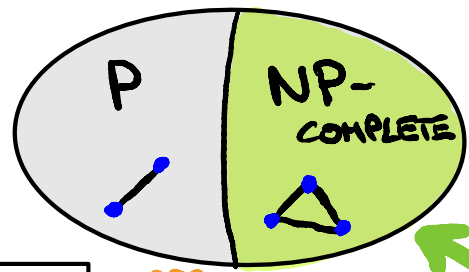
A FINITE



THEOREM

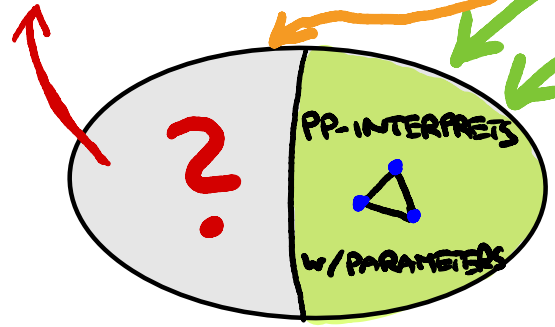
A FINITE

COMPLEXITY

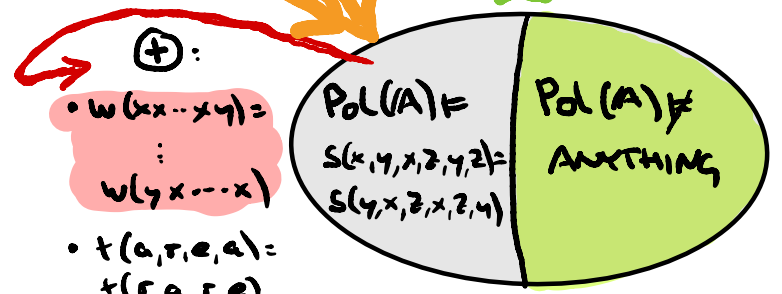


HELL + NESETRIL '91
 A UNDIRECTED
 \Rightarrow BIPARTITE OR LOOP

BARTO + KOZIK + NIVEN '07
 A SMOOTH, ALGEBRAIC LENGTH 1
 \Rightarrow LOOP



STRUCTURE



ALGEBRA

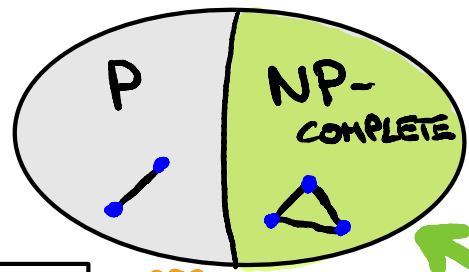
- \oplus :
- $w(x \dots x)$
- \vdots
- $w(y \dots x)$
- $t(a, r, e, a) = t(r, a, r, e)$
- MANY MORE



THEOREM

A FINITE

COMPLEXITY

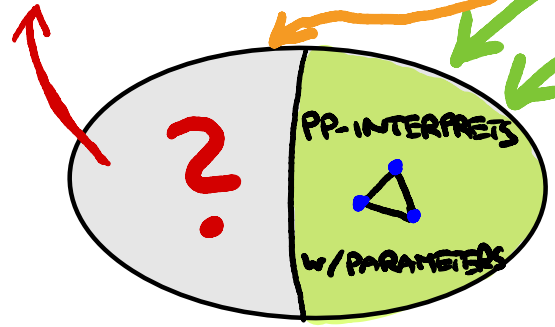


WANTED

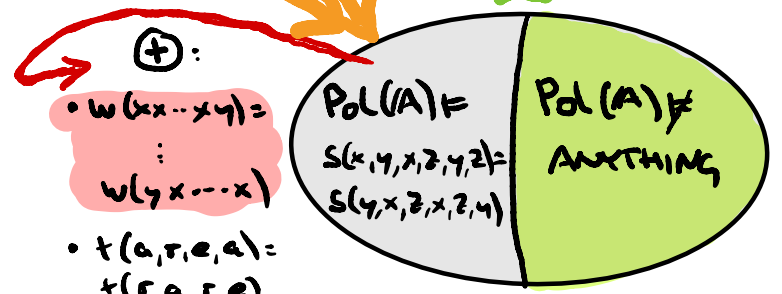
- ③ INFINITE \mathbb{A}
- ① STRUCTURE PROOFS
- ② STRUCTURE RESULTS
E.G. NO PARAMETERS

HELL + NEŠETŘIL '91
 \mathbb{A} UNDIRECTED
 \Rightarrow BIPARTITE OR LOOP

BARTO + KOZÍK + NIVEN '07
 \mathbb{A} SMOOTH, ALGEBRAIC LENGTH 1
 \Rightarrow LOOP



STRUCTURE



ALGEBRA

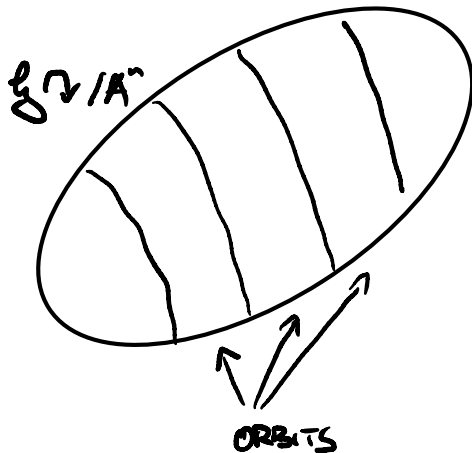
• MANY MORE



A INFINITE BUT

$\exists \varphi \in \text{Aut}(A):$

- $\forall n \ |A^n / \varphi^n|$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION



A INFINITE BUT

$\exists \varphi \in \text{Aut}(A):$

- $\forall n \ |A|_n / \varphi$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION

E.G. $(\mathbb{Q}, <)$

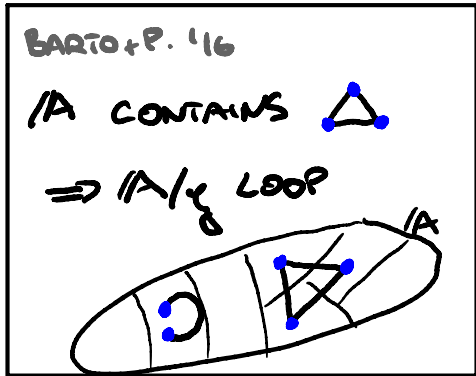
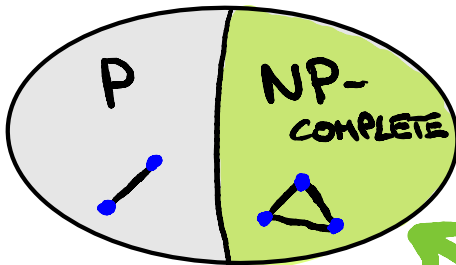
\mathbb{Q}
 F_0 -DEFINABLE
IN
FINITELY BOUNDED
HOMOGENEOUS
STRUCTURE

CONJECTURE (BODIRSKY + P. '11)

A INFINITE BUT $\exists \varphi \in \text{Aut}(A)$:

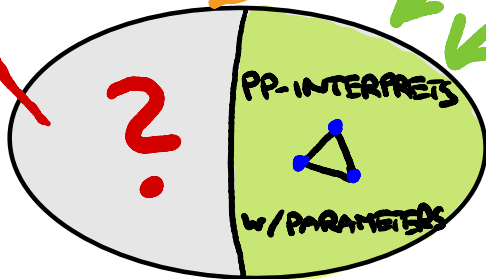
- $\forall n |A^n / \varphi|$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION

COMPLEXITY



\oplus

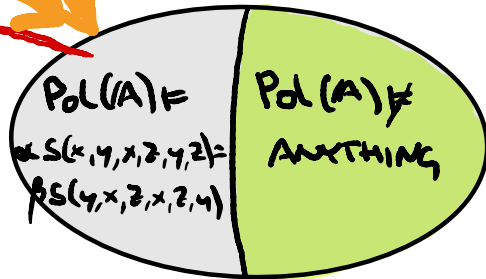
?



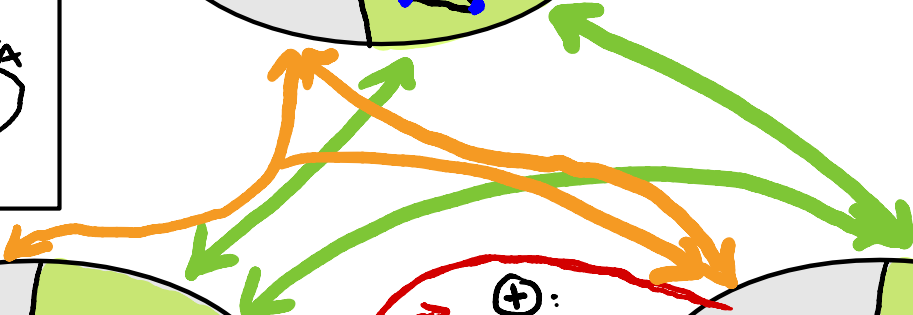
STRUCTURE

\oplus :

?



ALGEBRA



STRUCTURE

THEOREM "BETTER BARTO-KOZIK-NIVEN"

$\mathcal{G} \leq \text{Aut } A$, A FINITE

A SMOOTH, CONNECTED, ALGEBRAIC LENGTH 1


\exists PP-INTERPRETS  w/ \mathcal{G} -ORBITS

\Rightarrow LOOP 

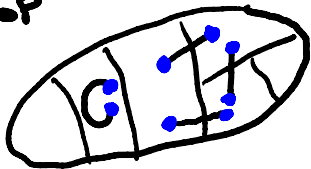
THEOREM "INFINITE HELL-NEŠETŘIL"

$\mathcal{G} \leq \text{Aut } A$, A/\mathcal{G} FINITE

A SMOOTH, A/\mathcal{G} SYMM., NON-BIPARTITE

\exists PP-INTERPRETS  w/ \mathcal{G} -ORBITS,
PARAMETERS

$\Rightarrow A/\mathcal{G}$ LOOP



STRUCTURE

THEOREM "BETTER BARTO-KOZIK-MVEN"

$\mathcal{L} \leq \text{Aut } A$, A FINITE

A SMOOTH, CONNECTED, ALGEBRAIC LENGTH 1


\exists PP-INTERPRETS  w/ \mathcal{L} -ORBITS

\Rightarrow LOOP 

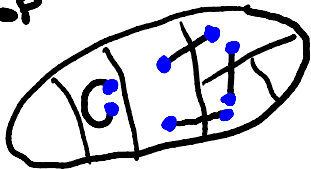
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$\mathcal{L} \leq \text{Aut } A$, A/\mathcal{L} FINITE

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\exists PP-INTERPRETS  w/ \mathcal{L} -ORBITS, PARAMETERS

$\Rightarrow A/\mathcal{L}$ LOOP



ALGEBRA

COROLLARY

A FINITE

\exists PP-INTERPRETS  (NO PARAMETERS)

$\Rightarrow \text{Pol}(A) \models S(\alpha_1 x, \dots, \alpha_n x, x, y, x, z, y, z) = S(y \dots y \ y \ x \ z \ x \ z \ y)$

COROLLARY

A/\mathcal{L} FINITE

$\exists A$ PP-INTERPRETS  w/ PARAMETERS

$\Rightarrow \text{Pol}(A) \models$ MANY THINGS

STRUCTURE

THEOREM "BETTER BARTO-KOZIK-MVEN"

$\mathcal{L} \leq \text{Aut } A$, A FINITE

A SMOOTH, CONNECTED, ALGEBRAIC LENGTH 1

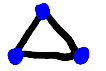
\exists PP-INTERPRETS  w/ \mathcal{L} -ORBITS

\Rightarrow LOOP 

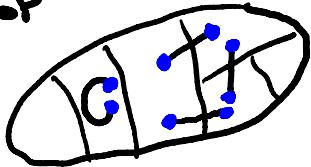
THEOREM "INFINITE HELL-NEŠETŘIL"

$\mathcal{L} \leq \text{Aut } A$, A/\mathcal{L} FINITE

A SMOOTH, A/\mathcal{L} SYMM., NON-BIPARTITE

\exists PP-INTERPRETS  w/ \mathcal{L} -ORBITS, PARAMETERS

$\Rightarrow A/\mathcal{L}$ LOOP



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COROLLARY

A/\mathcal{L} FINITE

$\exists A$ PP-INTERPRETS  w/ PARAMETERS

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COUNTEREXAMPLE

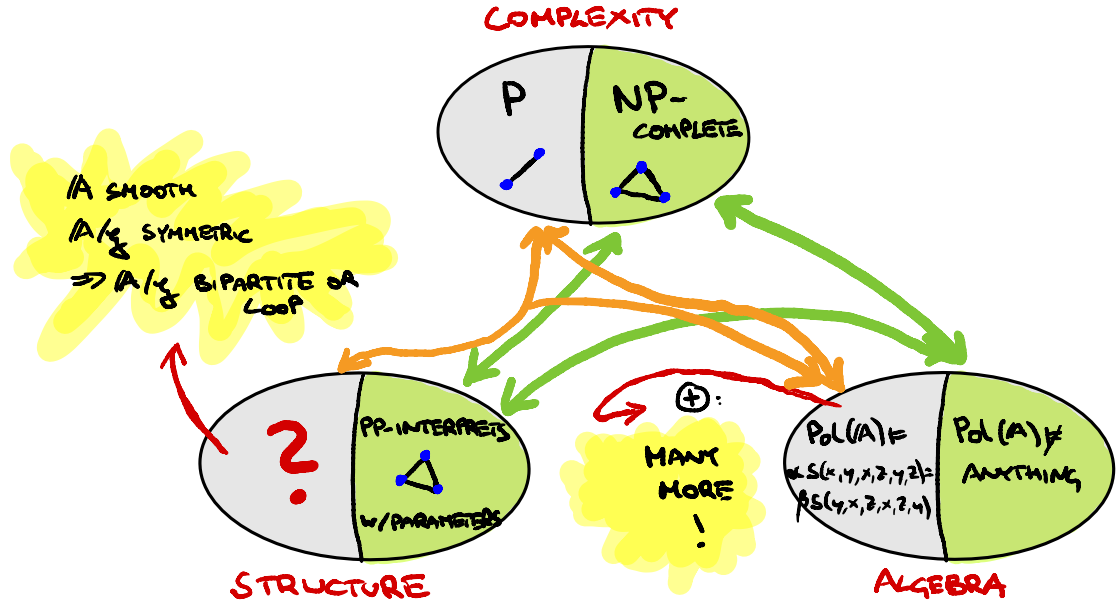
~~$\alpha_n w(x \dots x y) = \dots = \alpha_n w(y x \dots x)$~~

CONJECTURE (BODIRSKY & P. '11)

A INFINITE BUT $\exists \varphi \in \text{Aut}(A)$:

- $\forall n \ A^n / \varphi$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION

Thank you!



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FUNDED BY THE EUROPEAN UNION (ERC, POLCOOP, 101071674).

THERE WERE A COUPLE OF LIES IN THIS TALK.

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