CONSTRAINT SATISFACTION PROBLEMS

ALGEBRAIC & MODEL-THEORETIC CHALLENGES

DISTINGUISHING THE EASY FROM THE HARD

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COMBPROB LEEDS 11/2023

EUROPEAN RESEARCH COUNCIL

erc

ERC SYNERGY GRANT POCOCOP (GA 101071674) HO!

FWF I5948

OUTLINE

1. CONSTRAINT SATISFACTION PROBLEMS AND RELATIONAL STRUCTURES — 10 MINUTES
2. ALGEBRAIC INVARIANTS (FOR FINITE STRUCTURES) — 10 MINUTES
3. GOING INFINITE (ω-CATEGORICAL) — 10 MINUTES
4. GOING FINITE — 10 MINUTES

50 MINUTES

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1. CONSTRAINT SATISFACTION PROBLEMS AND RELATIONAL STRUCTURES — 10 MINUTES
2. ALGEBRAIC INVARIANTS (FOR FINITE STRUCTURES) — 10 MINUTES
3. GOING INFINITE (ω-CATEGORICAL) — 10 MINUTES
4. GOING FINITE — 10 MINUTES

~~50~~ 80 MINUTES

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$ SATISFYING ALL CONSTRAINTS ?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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MODEL:

- SET V OF POSSIBLE VALUES (FIXED)

  E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...

- ALLOWED CONSTRAINTS (FIXED)

  $C_1, \dots, C_m$ RELATIONS ON V

  $C_i \subseteq V^{a_i}$

SO $\mathbb{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE = "TEMPLATE"

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- PP-SENTENCE

$$\varphi = \exists x_1 \dots \exists x_n \; C_{i_1}(\text{VARIABLES}) \wedge C_{i_2}(\text{VARIABLES}) \wedge \dots \wedge C_{i_\ell}(\text{VARIABLES})$$

QUESTION $\mathbb{A} \models \varphi$?

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QUESTION $\mathbb{A} \models \varphi$?

META-QUESTION

FOR WHAT $\mathbb{A}$ IS CSP($\mathbb{A}$) EASY / HARD ?

P: $\exists$ ALGORITHM PROVIDING ANSWER IN p(# VARIABLES) STEPS, p POLYNOMIAL

#

NP: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLES

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

CSP($\mathbb{A}$)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1^2 + x_2^3 = x_3^4$

$\vdots$

QUESTION

- SOLUTION IN $\mathbb{Z}$?

<u>UNDECIDABLE</u> (MATIYASEVIC '77)

HILBERT 10

EXAMPLES

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- SAME OVER $\mathbb{Z}_p$:

NP-COMPLETE
$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

EXAMPLES

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- SAME WITHOUT $\cdot$:

IN P (GAUSS)

EXAMPLES

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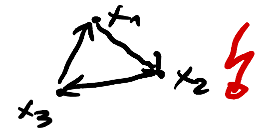
IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$) GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$
  $x_2 < x_3$
  $x_3 < x_1$
  $\vdots$

QUESTION
- SOLUTION IN $\mathbb{Q}$?



IN P

EXAMPLES

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

$\boxed{\text{CSP}(\mathbb{A})}$ $\boxed{\text{GIVEN}}$

- VARIABLES $x_1, \dots, x_n$
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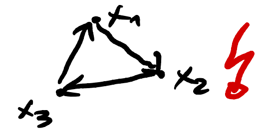
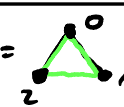


IN P

- $\mathbb{A} = K_3 =$  $= (\{0,1,2\}, E)$

$\boxed{\text{CSP}(\mathbb{A})}$ $\boxed{\text{GIVEN}}$

- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $E(x_1, x_2)$
  $E(x_2, x_3)$
  $\vdots$

$\boxed{\text{QUESTION}}$

- SOLUTION IN $\{0,1,2\}$?

3-COLORING PROBLEM = NP-COMPLETE

THEOREM (BULATOV, ZHUK '17)
(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

$\implies$ CSP($\mathbb{A}$) $\in$ P OR

NP-COMPLETE

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($P \neq NP$)

$CSP(\mathbb{A}) \in P \iff \mathbb{A}$ HAS ALGEBRAIC INVARIANT
$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

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ALGEBRAIC INVARIANTS = POLYMORPHISMS

$\mathbb{A} = (A; R_1, \ldots, R_m)$ STRUCTURE

$f: A^\ell \to A$ POLYMORPHISM $:\Leftrightarrow$

f HOMOMORPHISM $\mathbb{A}^\ell \to \mathbb{A}$ $\Leftrightarrow$

$\forall i: \forall t_1 \ldots t_\ell \in R_i$

$f(t_1 \ldots t_\ell) \in R_i$

$\boxed{\mathrm{Pol}(\mathbb{A})} := \{ f \mid f \text{ POLYMORPHISM OF } \mathbb{A} \}$

- CONTAINS PROJECTIONS $(x_1 \ldots x_\ell) \mapsto x_i$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS OF AN ALGEBRA ON A !

EXAMPLES

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$
- $(x,y,z) \mapsto x - y + z \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
- $? \in \mathrm{Pol}(K_3)$

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- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

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(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69)
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \Rightarrow \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\Rightarrow \mathrm{CSP}(\mathbb{A})$ HARDER THAN $\mathrm{CSP}(\mathbb{B})$

EXAMPLES

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- <u>GOOD</u> FOR CLASSIFYING CSPs ON $\{0,1\}$
  (SCHAEFER '78)

- <u>BETTER</u>: EQUATIONAL CONDITIONS:
  - $\exists f \in \mathrm{Pol}(\mathbb{Q}, <)\ \forall x,y \quad f(x,y) = f(y,x)$
  - $\exists f \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
    $\forall x,y,z \quad f(x,x,y) = f(y,x,x) = y$
  - $\exists f \in \mathrm{Pol}(K_3): \forall x \ fff(x) = x$

EXAMPLES

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$
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DEFINITION

- EQUATIONAL (STRONG MALCEV) CONDITION: SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots \quad s_1(\text{variables}) = t_1(\text{variables}) \wedge \dots \wedge s_k(\text{variables}) = t_k(\text{variables})$$

WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL :$\Leftrightarrow$ $\mathrm{Pol}(\text{ANY STRUCTURE}) \models \Psi$
  $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) := \{\Psi \mid \mathrm{Pol}(\mathbb{A}) \models \Psi\}$

(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KROKHIN '00)

$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

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(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

$\cancel{s}$ $\cancel{t}$

$$\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}^1(\mathrm{Pol}(\mathbb{B}))$$

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$\boxed{\text{COROLLARY}}$

$\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ CSP($\mathbb{A}$) NP-COMPLETE

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THEOREM $\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow$

$\mathrm{Pol}(\mathbb{A}) \models \exists f\ \forall x,y,z\quad f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

(SIGGERS '10)

$\models \exists f\ \forall a,r,e\quad f(a,r,e,a) = f(r,a,r,e)$

(KEARNES + MARKOVIĆ + MCKENZIE '14)

$\models \exists f\ \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall n \geq |A|$ PRIME

(BARTO + KOZIK '11)

$\models \exists f\ \forall x,y\quad f(x \dots xy) = \dots = f(yx \dots x)$

$\forall n \geq |A|$ PRIME

(MARÓTI + MCKENZIE '08)

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P (BULATOV, ZHUK '17)

GOING INFINITE

FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

E.G. 3-COLORING

INFINITE-DOMAIN CSPs:

EXAMPLES

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, \mathrm{Between}(x,y,z))$
- RANDOM GRAPH
- RANDOM POSET
- ATOMLESS BOOLEAN ALGEBRA

LOGIC PROBLEMS:

- $(\mathbb{Z}, +, 1, <)$
- $(\mathbb{Z}, +, \cdot)$
- $(\mathbb{Q}, +, \cdot)$

NUMERIC PROBLEMS

TEMPLATE = (COUNTABLE) ω-CATEGORICAL STRUCTURE

OFTEN: DEFINABLE IN HOMOGENEOUS STRUCTURE

"ARE THE CONSTRAINTS SATISFIABLE IN A LINEAR ORDER?" ETC.

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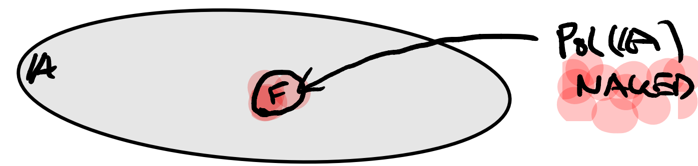
(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

$\mathbb{A}$ ω-CATEGORICAL

$EQ^{\wedge}_{\text{LOCAL}}(\mathrm{Pol}(\mathbb{A}))$ TRIVIAL $\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ CSP($\mathbb{A}$) NP-HARD

$\Leftrightarrow$ $\exists F \subseteq A$ FINITE: $EQ^{\wedge}(\mathrm{Pol}(\mathbb{A})|_F)$ TRIVIAL



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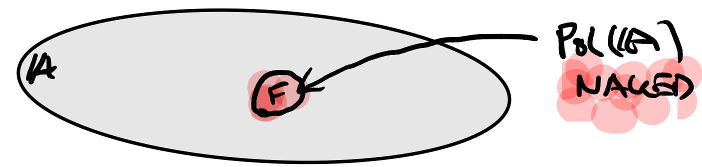

THEOREM

$\mathbb{A}$ ω-CATEGORICAL

$\underset{\text{LOCAL}}{\mathrm{EQ}^{\wedge}}(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow$

$\mathrm{Pol}(\mathbb{A}) \models \exists a, b, f\ \forall x, y, z$

$a \circ f(x,y,x,z,y,z) = b \circ f(y,x,z,x,z,y)$

(BARTO + P. '17)

F ... MANY MORE

(BARTO + BODOR + KOZIK + MOTTET + P. '23)

GOING ~~IN~~FINITE

$\mathrm{Pol}(\mathbb{A}) \models \exists u, v, f\ \forall x, y, z$

$$u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$$

... SO WHAT ?

PROBLEM: f MIGHT BE UGLY !

UGLINESS: IF $\mathbb{A}$ IS DEFINABLE IN $\mathbb{B}$
WHERE $\mathbb{B}$ IS HOMOGENEOUS,
d-ARY LANGUAGE

$\mathrm{CSP}(\mathbb{A})$: GIVEN $x_1 \dots x_n$
WANT TO ASSIGN $\mathrm{Aut}\,\mathbb{B}$-ORBITS
TO d-TUPLES OF VARIABLES

... BUT f MIGHT NOT ACT ON ORBITS

EXAMPLE $\mathrm{CSP}(\mathbb{Q}, \mathrm{Betw}(x, y, z))$

GIVEN VARIABLES $x_1, \dots, x_n$
WANT TO ASSIGN "<", ">", "="
TO EVERY PAIR (x_i, x_j)

GOING ~~IN~~FINITE

$\mathrm{Pol}(\mathbb{A}) \models \exists u, v, f \ \forall x, y, z$

$$u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$$

... SO WHAT ?

PROBLEM: f MIGHT BE UGLY !

UGLINESS: IF $\mathbb{A}$ IS DEFINABLE IN $\mathbb{B}$
WHERE $\mathbb{B}$ IS HOMOGENEOUS,
d-ARY LANGUAGE

CSP($\mathbb{A}$): GIVEN $x_1 \dots x_n$
WANT TO ASSIGN Aut $\mathbb{B}$-ORBITS
TO d-TUPLES OF VARIABLES

... BUT f MIGHT NOT ACT ON ORBITS

EXAMPLE CSP($\mathbb{Q}$, Betw(x,y,z))

GIVEN VARIABLES $x_1, \dots, x_n$
WANT TO ASSIGN "<", ">", "="
TO EVERY PAIR (x_i, x_j)

DEFINITION $\mathbb{B}$ HOMOGENEOUS, d-ARY

$f : B^\ell \to B$ CANONICAL WRT $\mathbb{B}$:$\Leftrightarrow$
f PRESERVES EQUIVALENCE RELATION
INDUCED BY Aut$(\mathbb{B}) \curvearrowright B^d$

EXAMPLE

- $f : \mathbb{Q} \to \mathbb{Q}$, $x \mapsto -x$ CANONICAL WRT $(\mathbb{Q}, <)$
- $\min : \mathbb{Q}^2 \to \mathbb{Q}$ NOT CANONICAL

$$\begin{array}{ccc} \min(a, b) & = & u \\ \vee\ \ \wedge & & ? \\ \min(c, d) & = & v \end{array}$$

GOING ~~IN~~FINITE

$\mathrm{Pol}(\mathbb{A}) \models \exists u,v,f\ \forall x,y,z$

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EXAMPLE $\mathrm{CSP}(\mathbb{Q}, \mathrm{Betw}(x,y,z))$

GIVEN VARIABLES $x_1, \dots, x_n$ WANT TO ASSIGN "<", ">", "=" TO EVERY PAIR (x_i, x_j)

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THEOREM (MOTTET + BODIRSKY '18)

$\mathbb{A}$ FO-DEFINABLE IN $\mathbb{B}$, $\mathbb{B}$ HOMOGENEOUS FINITELY BOUNDED

$f \in \mathrm{Pol}(\mathbb{A})$ CANONICAL WRT $\mathbb{B}$, ACTION ON ORBITS NON-TRIVIAL

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in P$

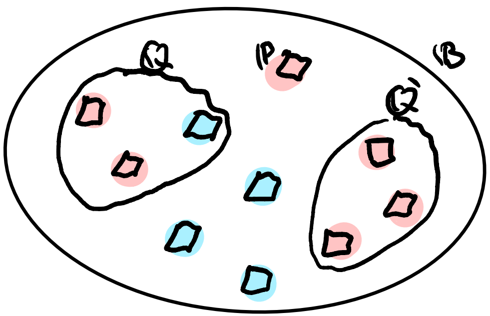
DEFINITION

$\mathbb{B}$ RAMSEY $:\Longleftrightarrow$

$\forall$ FINITE $\mathbb{P}, \mathbb{Q} \leq \mathbb{B}$

$\forall\ \chi : \binom{\mathbb{B}}{\mathbb{P}} \to 2$

$\exists\ \mathbb{Q}' \in \binom{\mathbb{B}}{\mathbb{Q}} : \chi|\binom{\mathbb{Q}'}{\mathbb{P}}$ CONSTANT



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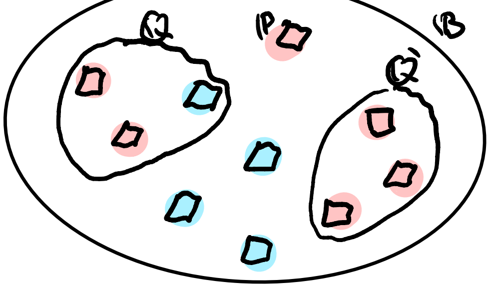

THEOREM (BODIRSKY + P. + TSANKOV '13)

$\mathbb{B}$ HOMOGENEOUS, d-ARY, RAMSEY

$\Rightarrow \forall f : B^e \to B$

$\exists\, \tilde{f} \in \overline{\{\alpha \circ f(\beta_1, \dots \beta_n) \mid \alpha, \beta_i \in \mathrm{Aut}(\mathbb{B})\}}$

$\tilde{f}$ CANONICAL

DEFINITION

$\mathbb{B}$ RAMSEY $:\Longleftrightarrow$

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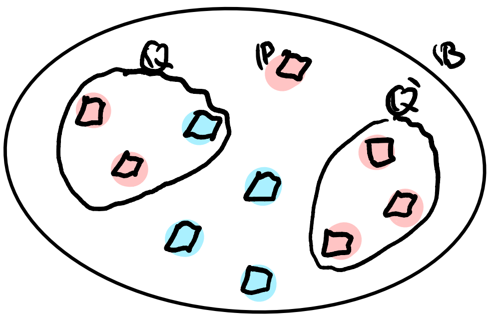

THEOREM (BODIRSKY + P. + TSANKOV '13)

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$\Rightarrow \forall f : \mathbb{B}^{\ell} \to \mathbb{B}$

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$\tilde{f}$ CANONICAL

THEOREM (MOTTET + P. '20)

$\mathrm{Pol}(\mathbb{A})$ NON TRIVIAL

BUT CANONICAL POLYMORPHISMS TRIVIAL

DEFINITION

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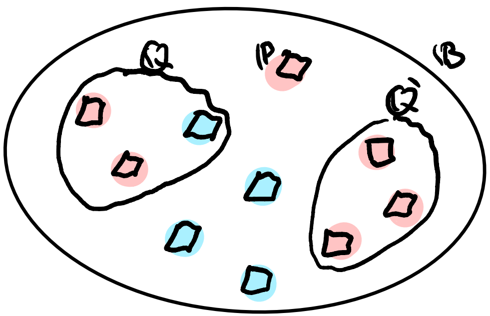

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$\Rightarrow \forall f : B^{\ell} \to B$

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THEOREM (MOTTET + P. '20)

$\mathrm{Pol}(\mathbb{A})$ NON TRIVIAL

BUT CANONICAL POLYMORPHISMS TRIVIAL

$\Rightarrow \exists s \in \mathrm{Pol}(\mathbb{A})$ ($\forall a, b$ ...

THEOREM

LET $\mathbb{A}$ BE FIRST-ORDER DEFINABLE IN:

- $\mathbb{Q}$ (BODIRSKY + KÁRA '07)
- THE RANDOM GRAPH (BODIRSKY + P. '11)
- ANY HOMOGENEOUS GRAPH (BODIRSKY + MARTIN + PONGRÁCZ + P. '16)
- THE UNIVERSAL HOMOGENEOUS TOURNAMENT (MOTTET + P. '20)
- PARTIAL ORDER (KOMPATSCHER + VAN PHAM '17)
- 2-BRANCHING C-RELATION (BODIRSKY + JONSSON + VAN PHAM '16)
- ANY FINITELY BOUNDED HOMOGENEOUS HYPERGRAPH WITH RAMSEY EXPANSION BY GENERIC TOTAL ORDER (MOTTET + NAGY + P. '23)

- IF $\mathrm{Pol}(\mathbb{A}) \models \exists u,v,f\ \forall x,y,z \quad u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$ THEN $\mathrm{CSP}(\mathbb{A}) \in P$
- OTHERWISE $\mathrm{CSP}(\mathbb{A})$ NP-COMPLETE

THEOREM

LET $\mathbb{A}$ BE FIRST-ORDER DEFINABLE IN:

- $\mathbb{Q}$ (BODIRSKY + KÁRA '07)
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  THEN $\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$
- OTHERWISE $\mathrm{CSP}(\mathbb{A})$ NP-COMPLETE

CONJECTURE (BODIRSKY + P. '11, BARTO + P. '16)

TRUE $\forall \mathbb{A}$ FO-DEFINABLE IN FINITELY BOUNDED HOMOGENEOUS $\mathbb{B}$

THANK YOU !

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