LOOP CONDITIONS

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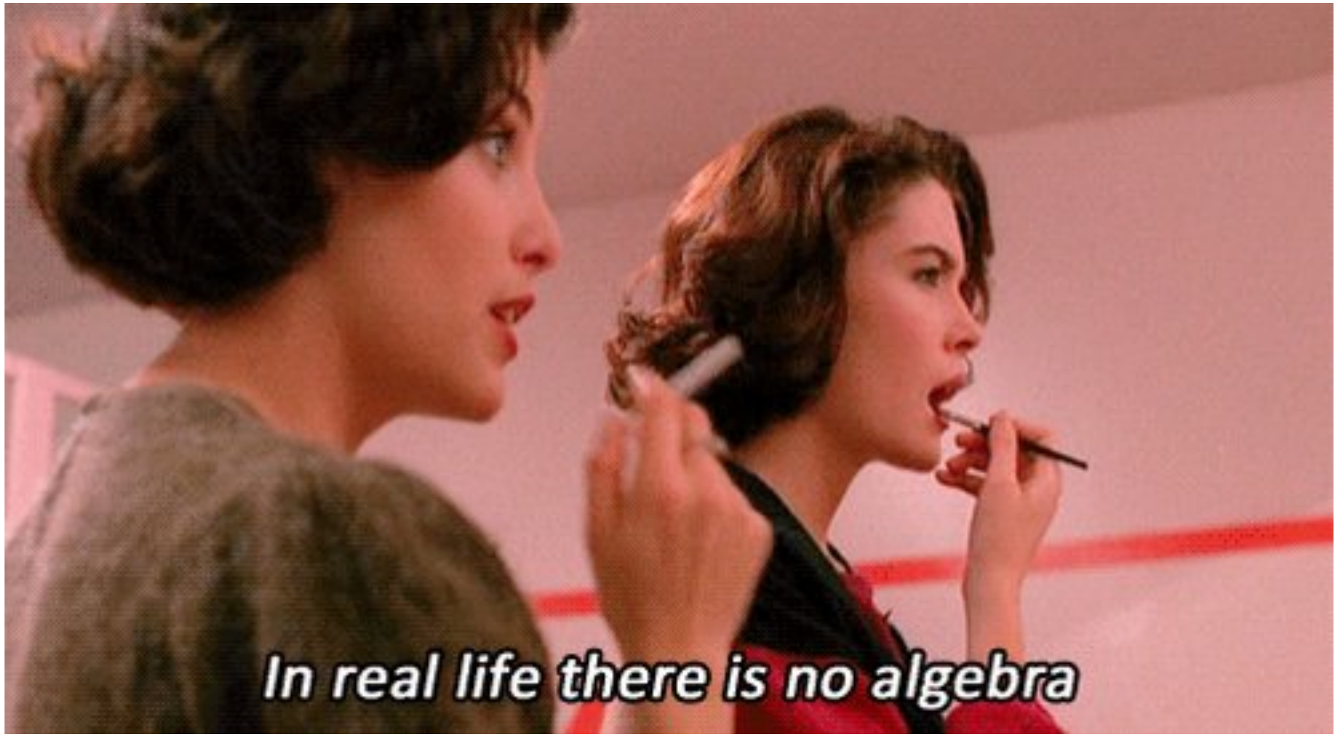
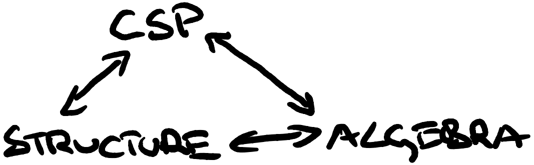
FWF I5948

HO!

ALGEBRA WEEK
SIENA 2023

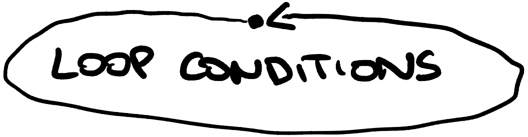

In real life there is no algebra

PART I



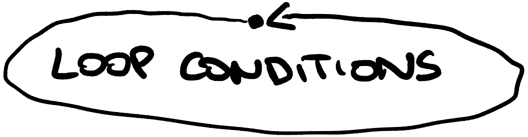
- OVERVIEW
- RECENT PROGRESS

PART II



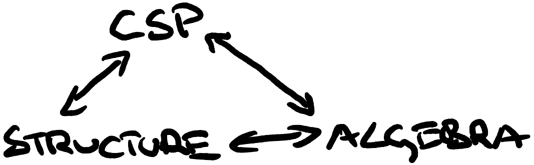
- DEFINITION
- EXAMPLES
- $$\begin{tikzcd} \text{ALGEBRA} \arrow[r] & \text{STRUCTURE} \arrow[d] \\ & \text{CSP} \end{tikzcd}$$

PART III



- STRUCTURE → ALGEBRA

PART I

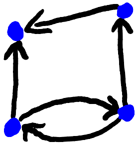


- OVERVIEW
- RECENT PROGRESS

JOINT WITH : LIBOR BARTO
BERTALAN BODOR
MARCIN KOZIK
ANTOINE MOTTET

THIS PART CONTAINS A COUPLE OF LIES

$\mathbb{A}$ ... DIGRAPH / RELATIONAL STRUCTURE (POSSIBLY INFINITE)

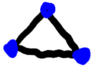


CSP($\mathbb{A}$)

GIVEN:
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 \to x_2$, $x_2 \to x_3$, $x_3 \to x_1$, ...

QUESTION: $\exists$ SOLUTION IN $\mathbb{A}$?

EXAMPLES

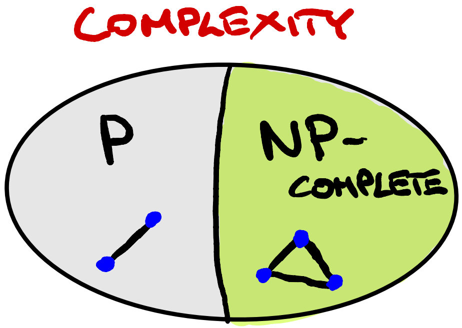
- $\mathbb{A} =$  3-COLORING
- $\mathbb{A} =$  2-COLORING
- $\mathbb{A} = (\mathbb{Q}, <) =$  DIGRAPH ACYCLICITY
- $\mathbb{A} = (\mathbb{Q}, \text{BETWEEN}(x,y,z)) =$  BETWEENNESS PROBLEM

COMBINATORIAL PROBLEMS

LOGIC PROBLEMS
E.G. MMSNP
MODEL CHECKING

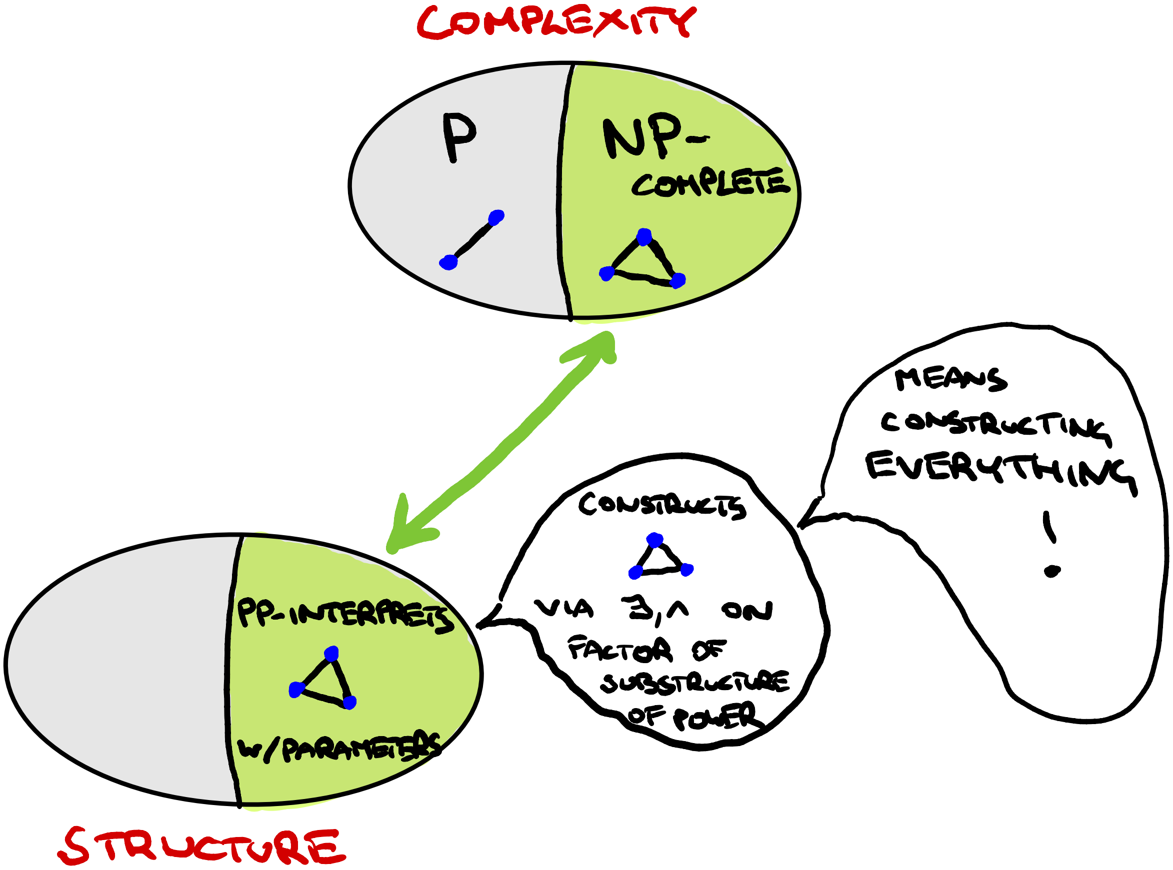
THEOREM (BULATOV, ZHUK '17 – CONJECTURE OF FEDER+VARDI 90s)

$\mathbb{A}$ FINITE



THEOREM

$\mathbb{A}$ FINITE



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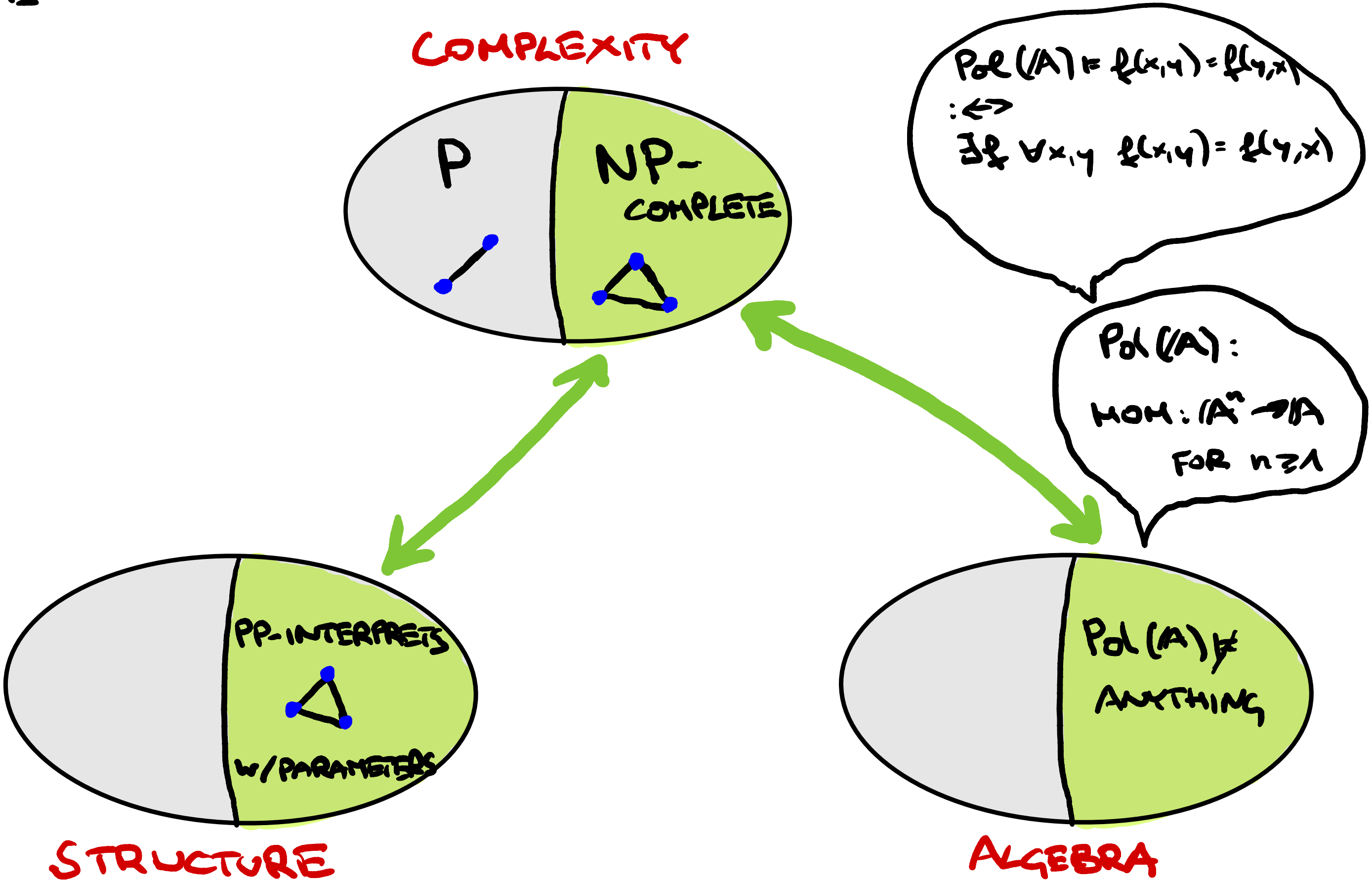

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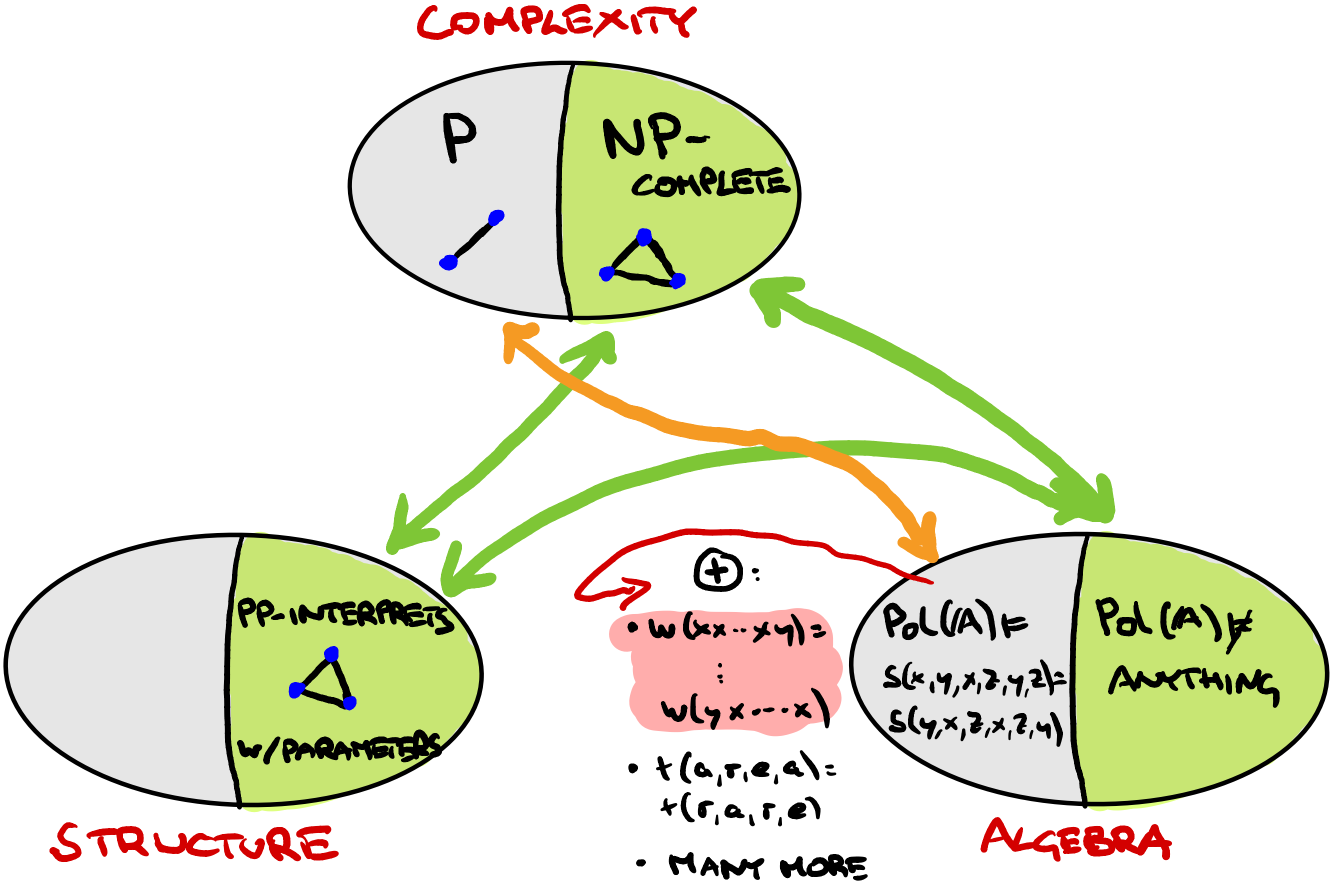
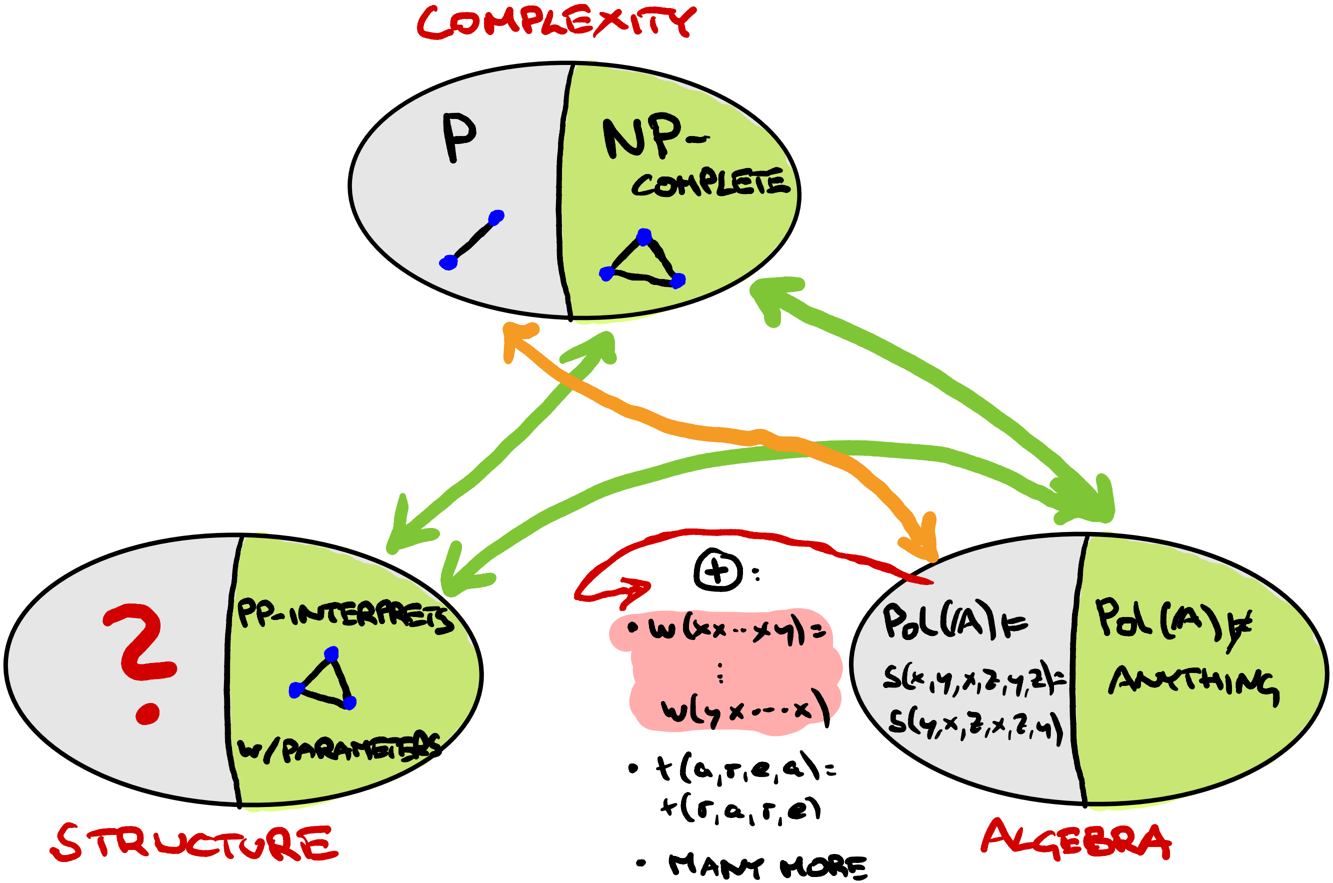


⊕:

- w(xx⋯xy) = ⋮ w(yx⋯x)
- t(a,r,e,a) = t(r,a,r,e)
- MANY MORE

THEOREM

$\mathbb{A}$ FINITE



THEOREM

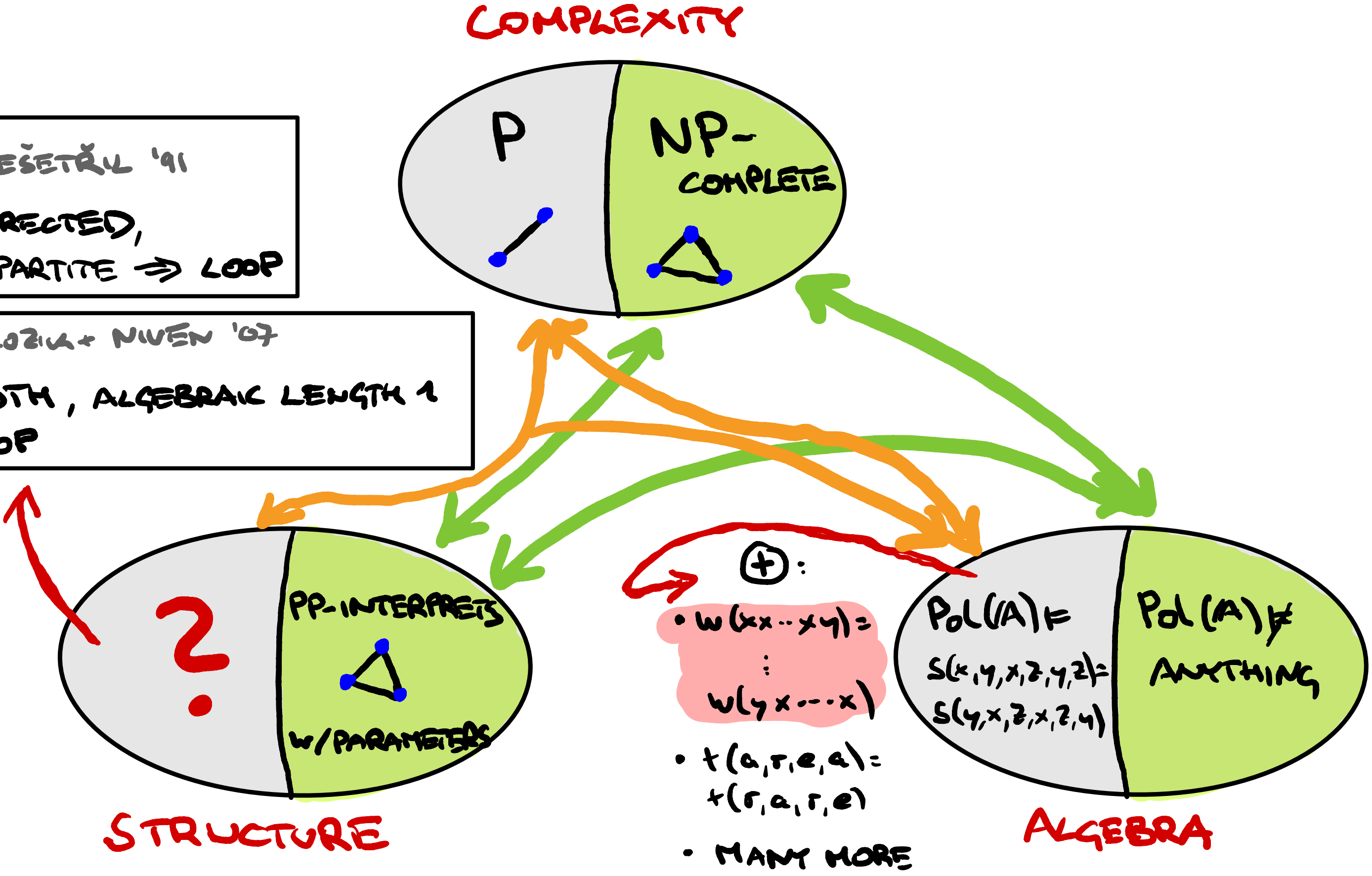
$\mathbb{A}$ FINITE

HELL + NEŠETŘIL '91

$\mathbb{A}$ UNDIRECTED, NON-BIPARTITE ⇒ LOOP

BARTO + KOZIK + NIVEN '07

$\mathbb{A}$ SMOOTH, ALGEBRAIC LENGTH 1 ⇒ LOOP



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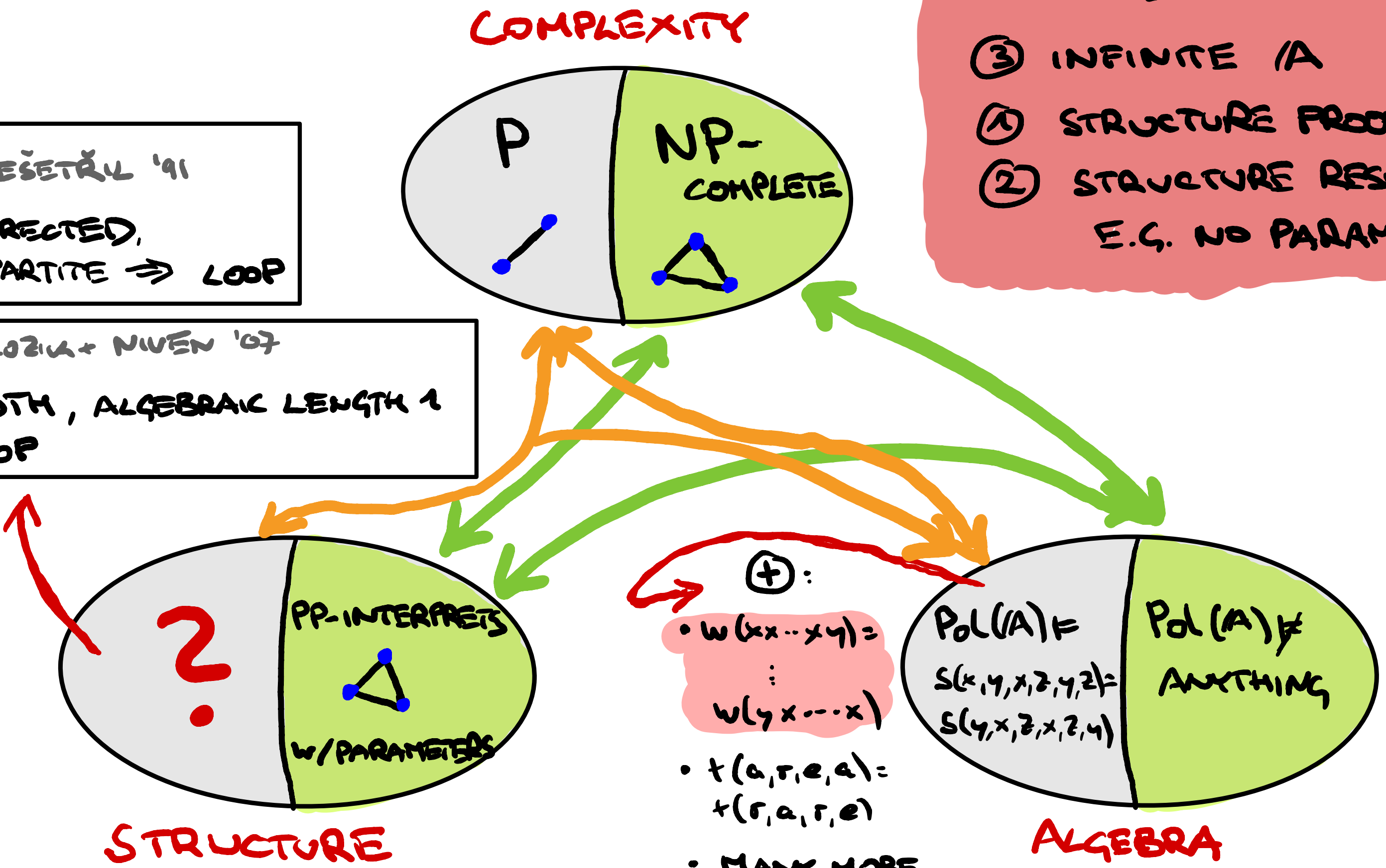
WANTED

③ INFINITE $\mathbb{A}$

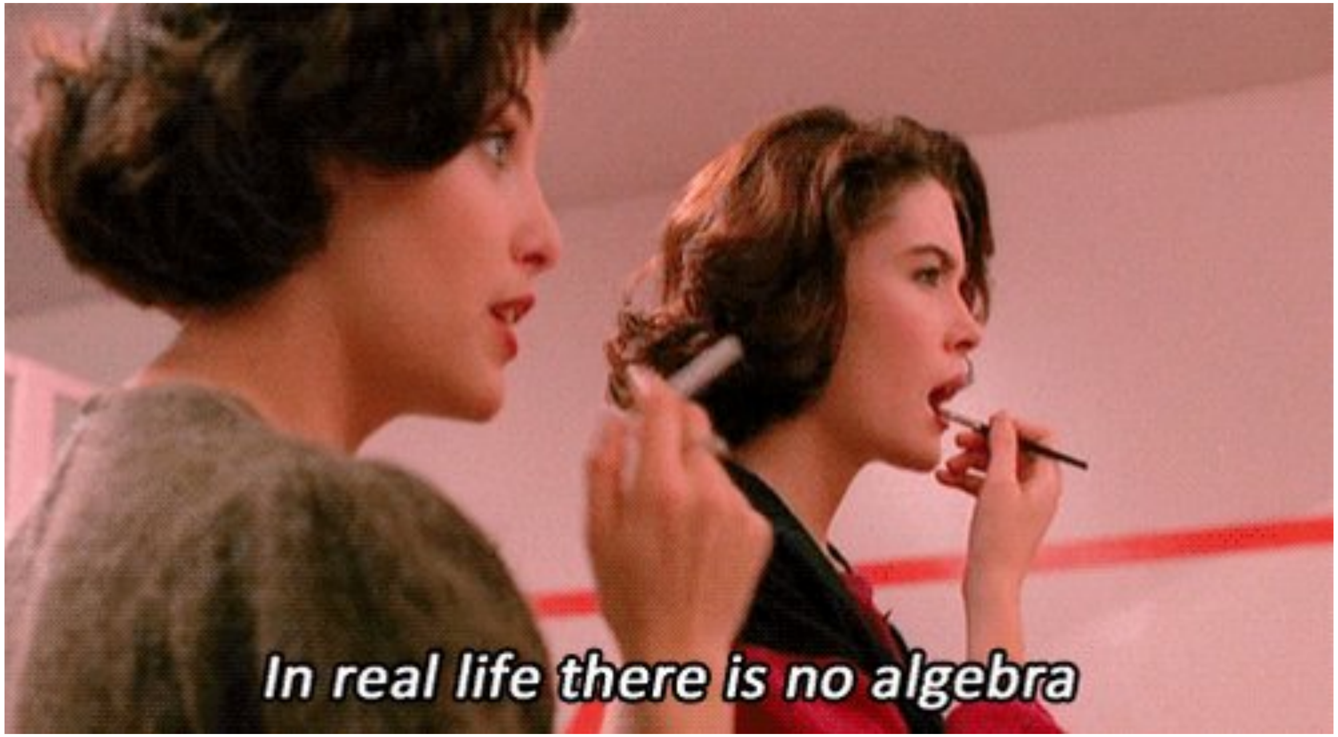
① STRUCTURE PROOFS

② STRUCTURE RESULTS
E.G. NO PARAMETERS



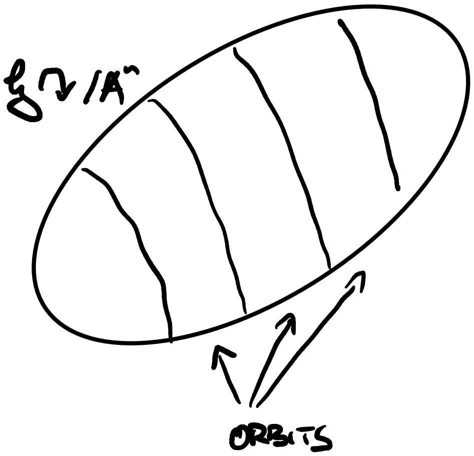
⊕:

- $w(xx\cdots xy) =$
  $\vdots$
  $w(yx\cdots x)$
- $t(a,r,e,a) = t(r,a,r,e)$
- MANY MORE


In real life there is no algebra

$\mathbb{A}$ INFINITE BUT $\exists\, \mathcal{G} \leq \mathrm{Aut}(\mathbb{A})$:

- $\forall n$ $\mathbb{A}^n/\mathcal{G}$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION



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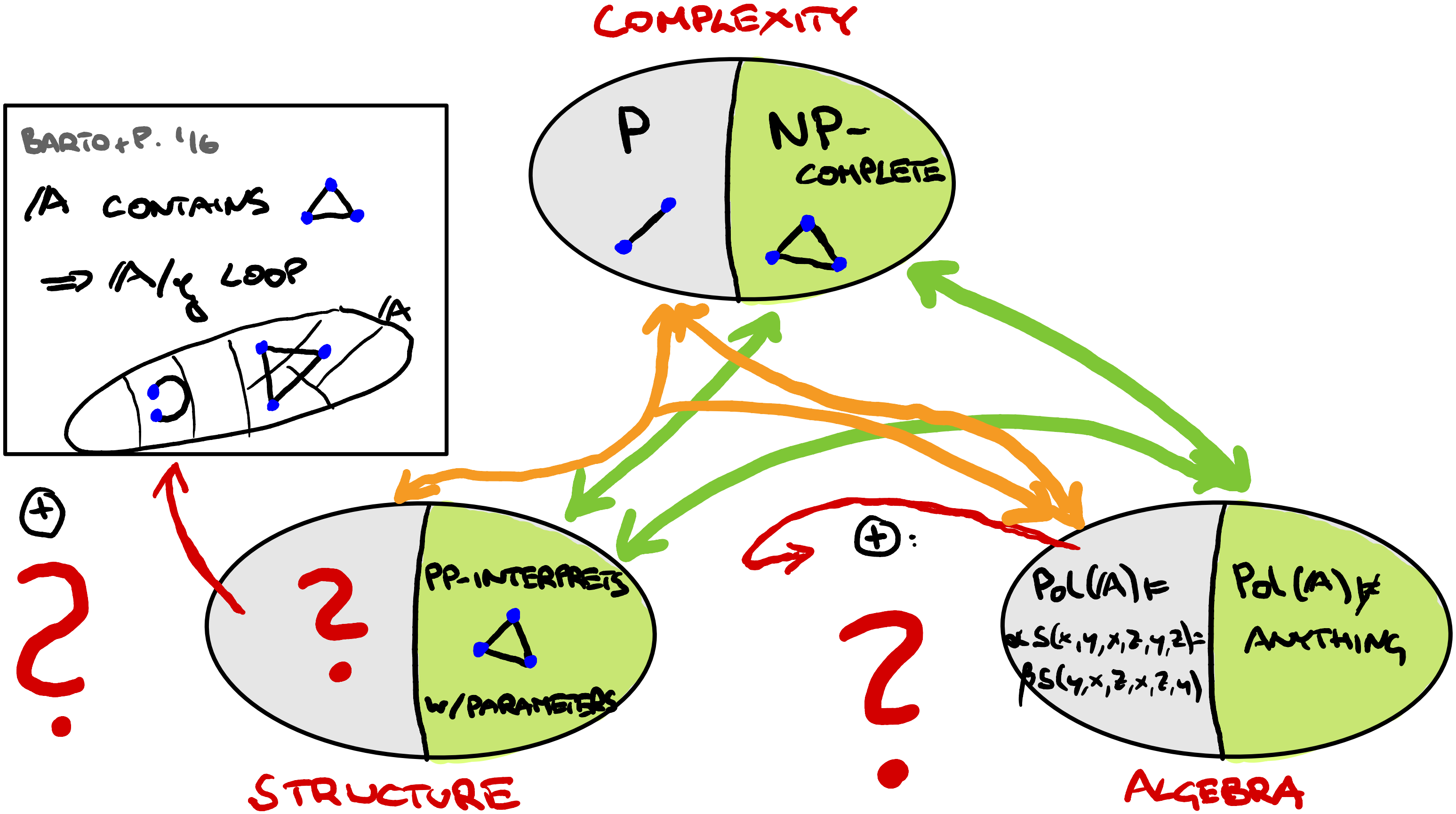
E.G. $(\mathbb{Q},<)$

$\mathbb{A}$
FO-DEFINABLE
IN
FINITELY BOUNDED
HOMOGENEOUS
STRUCTURE

CONJECTURE (BODIRSKY + P. '11)

$\mathbb{A}$ INFINITE BUT $\exists \mathcal{G} \leq Aut(\mathbb{A})$: $\left\{ \begin{array}{l} \bullet\ \forall n\ \mathbb{A}^n/\mathcal{G} \text{ FINITE (“ORBITS”)} \\ \bullet\ \text{ORBITS HAVE EFFECTIVE DESCRIPTION} \end{array} \right.$

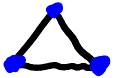


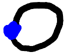
STRUCTURE

THEOREM „BETTER BARTO-KOZIK-NIVEN"

$\mathcal{G} \leq \mathrm{Aut}\,\mathbb{A}$, $\mathbb{A}$ FINITE

$\mathbb{A}$ SMOOTH, CONNECTED, ALGEBRAIC LENGTH 1

$\neg$ PP-INTERPRETS  w/ $\mathcal{G}$-ORBITS

$\Rightarrow$ LOOP 

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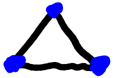
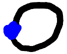
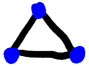
$\neg$ PP-INTERPRETS  W/ $\mathcal{G}$-ORBITS

$\Rightarrow$ LOOP 

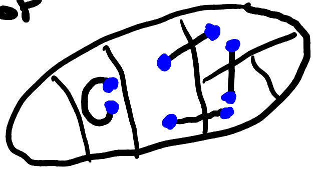
THEOREM „INFINITE HELL-NEŠETŘIL“

$\mathcal{G} \leq \operatorname{Aut} \mathbb{A}$, $\mathbb{A}/\mathcal{G}$ FINITE

$\mathbb{A}$ SMOOTH, $\mathbb{A}/\mathcal{G}$ SYMM., NON-BIPARTITE

$\neg$ PP-INTERPRETS  W/ $\mathcal{G}$-ORBITS, PARAMETERS

$\Rightarrow \mathbb{A}/\mathcal{G}$ LOOP



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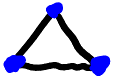
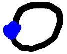
$\neg$ PP-INTERPRETS  W/ $\mathcal{G}$-ORBITS

$\Rightarrow$ LOOP 

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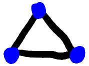
$\neg$ PP-INTERPRETS  W/ $\mathcal{G}$-ORBITS, PARAMETERS

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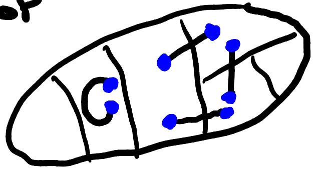
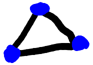


ALGEBRA

COROLLARY

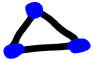
$\mathbb{A}$ FINITE

$\neg$ PP-INTERPRETS  (NO PARAMETERS)

$\Rightarrow \mathrm{Pol}(\mathbb{A}) \models s(\alpha_1 x, \dots, \alpha_m x, x, y, x, z, y, z) = s(y \ \dots \ y \ y \ x \ z \ x \ z \ y)$

COROLLARY

$\mathbb{A}/\mathcal{G}$ FINITE

$\neg \mathbb{A}$ PP-INTERPRETS  W/ PARAMETERS

$\Rightarrow \mathrm{Pol}(\mathbb{A}) \models$ MANY THINGS

STRUCTURE

THEOREM "BETTER BARTO-KOZIK-NIVEN"

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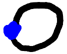
$\mathbb{A}$ SMOOTH, CONNECTED, ALGEBRAIC LENGTH 1

$\neg$ PP-INTERPRETS △ w/ $\mathcal{G}$-ORBITS

$\Rightarrow$ LOOP



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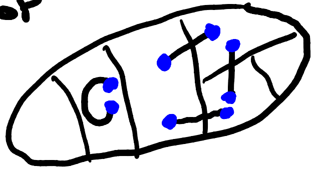

ALGEBRA

COROLLARY

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$$\Rightarrow \mathrm{Pol}(\mathbb{A}) \models s(\alpha_1 x, \dots, \alpha_m x, x, y, x, z, y, z) = s(y \cdots y\ y\ x\ z\ x\ z\ y)$$

COROLLARY

$\mathbb{A}/\mathcal{G}$ FINITE

$\neg\mathbb{A}$ PP-INTERPRETS △ w/ PARAMETERS

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A}) \models$ MANY THINGS

COUNTEREXAMPLE

$$\xcancel{\alpha_1 w(x\cdots xy) = \cdots = \alpha_n w(yx\cdots x)}$$

CONJECTURE (BODIRSKY + P. '11)

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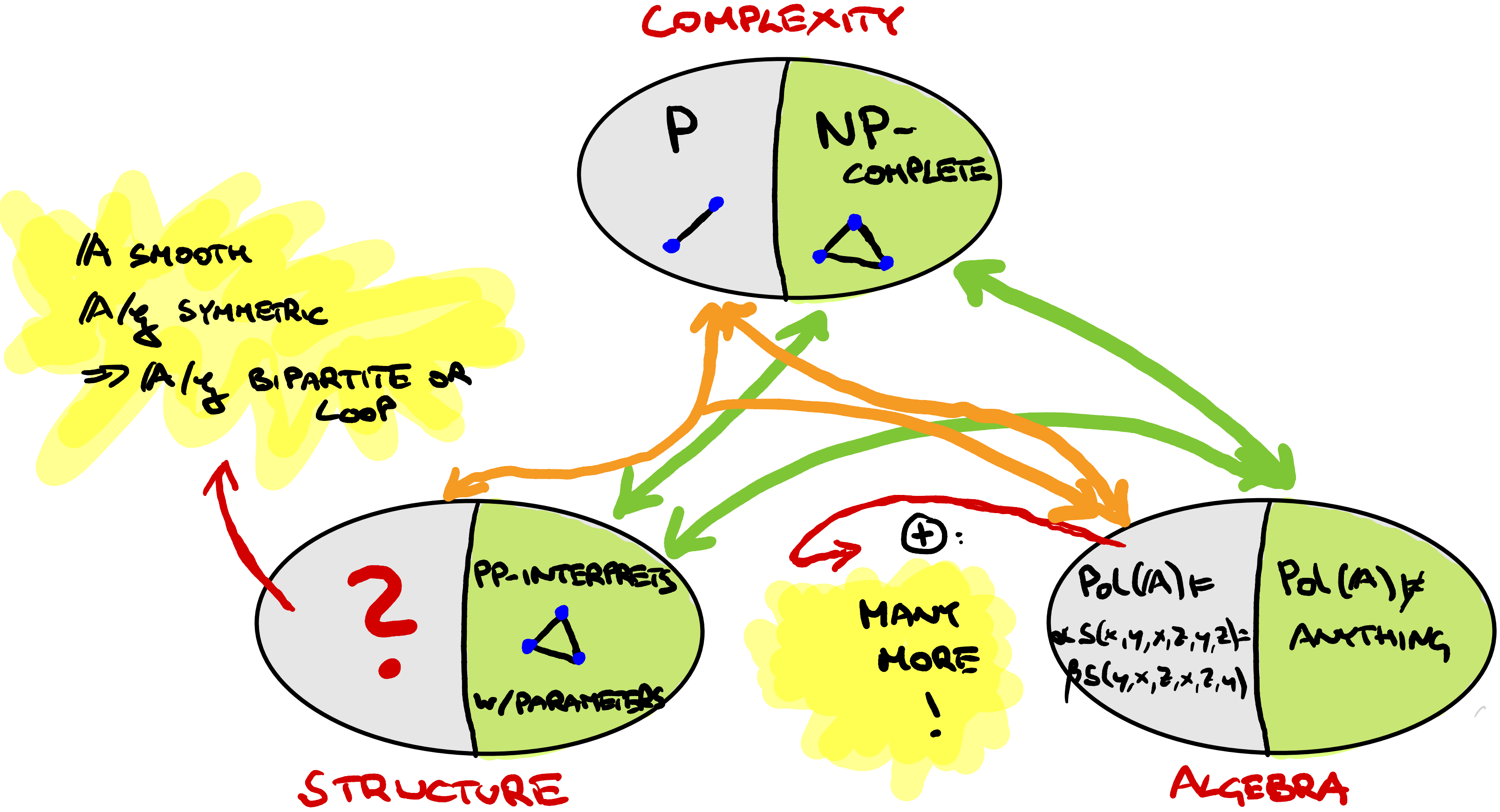


PART II

LOOP CONDITIONS

- DEFINITION
- EXAMPLES
- ALGEBRA → STRUCTURE ↓ CSP

THIS PART SHOULD ACTUALLY BE CORRECT

LOOP CONDITION :

- STATEMENT OF THE FORM :

WHENEVER R IS A RELATION SATISFYING (C), THEN R HAS A LOOP ($\exists c\ (c, \ldots, c) \in R$) .

LOOP CONDITION:

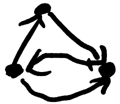
- STATEMENT OF THE FORM:

  WHENEVER R IS A RELATION SATISFYING (C),
  THEN R HAS A LOOP ($\exists c \ (c,\dots,c) \in R$).

- OFTEN: $(C) = (S) \wedge (A)$ WHERE:

(S) ... STRUCTURAL CONDITION

E.G. "R IS A NON-BIPARTITE GRAPH"

"R IS A DIGRAPH CONTAINING  "

"R IS SYMMETRIC: $\forall \sigma \in S_n \ \forall (a_1 \dots a_n) \in R$
$(a_{\sigma 1} \dots a_{\sigma n}) \in R$ "

(A) ... ALGEBRAIC CONDITION

E.G. " R IS INVARIANT UNDER A TAYLOR OPERATION"
" ... MAJORITY ... "

EXAMPLE

WHENEVER R IS (S) SYMMETRIC, BINARY (I.E. A GRAPH)

$\wedge$ (A) $\exists s \in \mathrm{Pol}(R)$ 6-ARY SUCH THAT

$\forall x,y,z \quad s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

⌈ "SIGGERS POLYMORPHISM" ⌋

$\Rightarrow$ R HAS LOOP.

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- (COUNTER) EXAMPLE :

INDEPENDENT SET

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TRUE OR FALSE ?

- **(COUNTER) EXAMPLE** :

  INDEPENDENT SET

- **(COUNTER) EXAMPLE** :

  SINGLE EDGE 

  $\mathrm{maj}_7(x_1,\dots,x_7) \in \mathrm{Pol}(R)$:

$$\Rightarrow \mathrm{maj}_7(a_1 \dots a_7) = 1$$

$$\mathrm{maj}_7(b_1 \dots b_7) = 0$$

$$s(x_1,\dots,x_6) := \mathrm{maj}_7(x_1 \dots x_6 x_6)$$

$$\Rightarrow s(x,y,x,z,y,z) = \mathrm{maj}_7(x,y,x,z,y,z,z)$$

$$= \mathrm{maj}_3(x,y,z)$$

$$s(y,x,z,x,z,y) = \mathrm{maj}_3(x,y,z)$$

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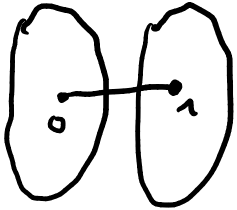
$\Rightarrow$ R HAS LOOP.

(COUNTER)EXAMPLE:

R BIPARTITE:



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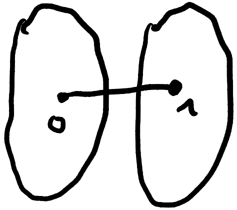

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⌜"SIGGERS POLYMORPHISM"⌟

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(COUNTER)EXAMPLE:

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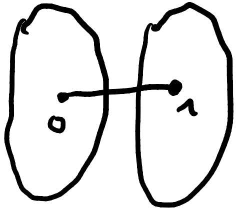

~~(COUNTER)~~ EXAMPLE:

BABY-LOOP CONDITION: $\overbrace{\text{E NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\text{Pol(E) HAS SIGGERS}}^{(A)}$

$\Rightarrow$ E HAS LOOP

WHENEVER R IS (S) SYMMETRIC, BINARY (i.E. A GRAPH)

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⌜"SIGGERS POLYMORPHISM"⌟

$\Longrightarrow$ R HAS LOOP.

(COUNTER)EXAMPLE:

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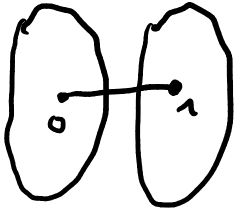


(~~COUNTER)~~ EXAMPLE:

BABY-LOOPCONDITION: $\overbrace{E \text{ NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\mathrm{Pol}(E) \text{ HAS SIGGERS}}^{(A)}$

$\Longrightarrow$ E HAS LOOP

PROOF WE SHOW $\underset{\text{ODD}}{\ell \geq 1}$, $\underbrace{C_\ell}_{\text{UNDIRECTED CYCLE OF LENGTH } \ell} \xrightarrow{\text{HOM}} E \Longrightarrow E$ HAS LOOP $\Big\}$ SUFFICIENT

WHENEVER R IS (S) SYMMETRIC, BINARY (i.e. A GRAPH)

$\wedge$ (A) $\exists s \in \mathrm{Pol}(R)$ 6-ARY SUCH THAT

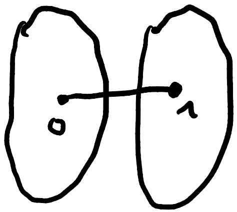
$\forall x,y,z \quad s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

⌜"SIGGERS POLYMORPHISM"⌟

$\Rightarrow$ R HAS LOOP.

(COUNTER)EXAMPLE:

R BIPARTITE:



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- $\ell = 1$ ✓

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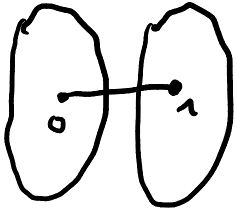

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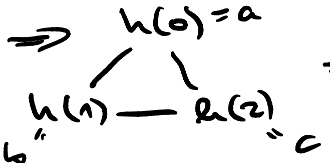
PROOF WE SHOW $\ell \geq 1$ ODD, $\underset{\text{UNDIRECTED CYCLE OF LENGTH } \ell}{C_\ell} \xrightarrow{HOM} E \Rightarrow E$ HAS LOOP } SUFFICIENT

- $\ell = 1$ ✓
- $\ell = 3$ $\quad h: C_3 \to G$

$\Rightarrow$



$\Rightarrow$

$$\left.\begin{array}{l} s(a,b,a,c,b,c) =: u \\ \quad | \quad | \quad | \quad | \quad | \quad | \\ s(b,a,c,a,c,b) =: u \end{array}\right)$$

WHENEVER R IS (S) SYMMETRIC, BINARY (i.e. A GRAPH)

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⌜"SIGGERS POLYMORPHISM"⌟

⇒ R HAS LOOP.

(COUNTER)EXAMPLE:

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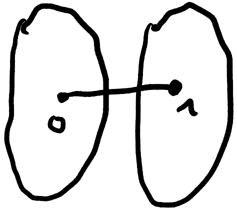

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BABY-LOOPCONDITION: $\overbrace{\text{E NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{Pol(E) \text{ HAS SIGGERS}}^{(A)}$

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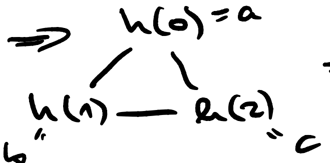
PROOF WE SHOW $\ell \geq 1$ ODD, $C_\ell \xrightarrow{HOM} E$ ⇒ E HAS LOOP } SUFFICIENT

UNDIRECTED CYCLE OF LENGTH ℓ

- $\ell = 1$ ✓
- $\ell = 3$ $\quad h: C_3 \rightarrow G$

⇒



⇒ $\left.\begin{array}{c} s(a,b,a,c,b,c) =: u \\ | \; | \; | \; | \; | \; | \\ s(b,a,c,a,c,b) =: u \end{array}\right)$

- $\ell = 3^n$, $n \geq 2$

SET $E_n := \underbrace{E \circ \dots \circ E}_{3^{n-1}}$

$C_{3^n} \rightarrow E \Rightarrow C_3 \rightarrow E_n$

$\Rightarrow E_n$ LOOP $\Rightarrow C_{3^{n-1}} \rightarrow E$

$\dots \quad \Rightarrow C_3 \rightarrow E \Rightarrow$ LOOP □

BABY-LOOP CONDITION: $\overbrace{\text{E NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\text{POL(E) HAS SIGGERS}}^{(A)}$

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BABY-LOOP CONDITION: $\overbrace{E \text{ NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{Pol(E) \text{ HAS SIGGERS}}^{(A)}$

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THEOREM (HELL + NEŠETŘIL '91): E $\overbrace{\text{FINITE NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{Pol(E) \text{ TAYLOR}}^{(A)}$

(INFANT)

$\Rightarrow E$ HAS LOOP

$Pol(E) \models$

$t(x, ?, \dots, ?) = t(y, ?, \dots ?)$

$t(?, x, ? \dots ?) = t(?, y, ?, \dots ?)$

$\vdots$

BABY-LOOP CONDITION: $\overbrace{\text{E NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\text{Pol(E) HAS SIGGERS}}^{(A)}$

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$\Rightarrow$ E HAS LOOP

(INFANT)

$$\mathrm{Pol}(E) \models \bigwedge \begin{array}{l} t(x, ?, \dots, ?) = t(y, ?, \dots, ?) \\ t(?, x, ?, \dots, ?) = t(?, y, ?, \dots, ?) \\ \vdots \end{array}$$

REALLY: $\left.\begin{array}{l}\overbrace{\text{E FINITE NON-BIPARTITE GRAPH,}}^{(S)} \\ \text{DOES NOT PP-INTERPRET } K_3 \text{ W/ PAR.}\end{array}\right\} \Rightarrow$ E HAS LOOP

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$\text{Pol}(E) \models$

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$\wedge\ t(?,x,?\dots?) = t(?,y,?,\dots)$

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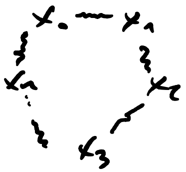
(ADOLESCENT)



$\sim\ \in \{\rightarrow, \leftarrow\}$

$(\# \rightarrow) - (\# \leftarrow) = 1$

ADULT STUFF :

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THEOREM (MAROTI + MCKENZIE '07)

$$\overbrace{A \text{ FINITE}, \; R \subseteq A^n \text{ SYMMETRIC}, \; n > |A| \text{ PRIME}}^{(S)}, \; \overbrace{\mathrm{Pol}(R) \text{ TAYLOR}}^{(A)}$$

$$\implies R \text{ LOOP}$$

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THEOREM (BARTO + KOZIK '11)

SYMMETRIC $\rightsquigarrow$ CYCLICALLY SYMMETRIC :

$$\forall (a_1, \dots a_n) \in R \quad (a_2, \dots a_n a_1) \in R$$

ADULT STUFF:

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$\vdots$

MANY MORE: OLŠÁK, BARTO + P., GILLIBERT + JONUŠAS + P., MOTTET + P., ZHUK,
BARTO + BODOR + KOZIK + MOTTET + P.

MEANING FOR CSP

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THEOREM (HELL + NEŠETŘIL '91):

$$\left.\overbrace{\begin{array}{l}\text{E FINITE NON-BIPARTITE GRAPH,}\\ \text{DOES NOT PP-INTERPRET } K_3 \text{ W/PAR.}\end{array}}^{(S)}\right\} \Rightarrow \text{E HAS LOOP}$$

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COROLLARY E FINITE GRAPH ⇒ CSP(E) IN P OR NP-COMPLETE

MEANING FOR CSP

THEOREM (HELL + NEŠETŘIL '91):

$\overbrace{\text{E FINITE NON-BIPARTITE GRAPH,}}^{(s)}$
DOES NOT PP-CONSTRUCT K_3 } ⟹ E HAS LOOP

COROLLARY E FINITE GRAPH ⟹ CSP(E) IN P OR NP-COMPLETE

PROOF:
- E PP-INTERPRETS K_3 W/PAR ⟹ CSP(E) NP-COMPLETE (EASY)
- OTHERWISE, E BIPARTITE OR LOOP SO CSP(E) = 2-COLORING OR TRIVIAL

MEANING FOR CSP

THEOREM (HELL + NEŠETŘIL '91):

(S)

$\left.\begin{array}{l}\text{E FINITE NON-BIPARTITE GRAPH,} \\ \text{DOES NOT PP-CONSTRUCT } K_3\end{array}\right\} \Rightarrow$ E HAS LOOP

COROLLARY E FINITE GRAPH ⇒ CSP(E) IN P OR NP-COMPLETE

PROOF:
- E PP-INTERPRETS K_3 W/PAR ⇒ CSP(E) NP-COMPLETE (EASY)
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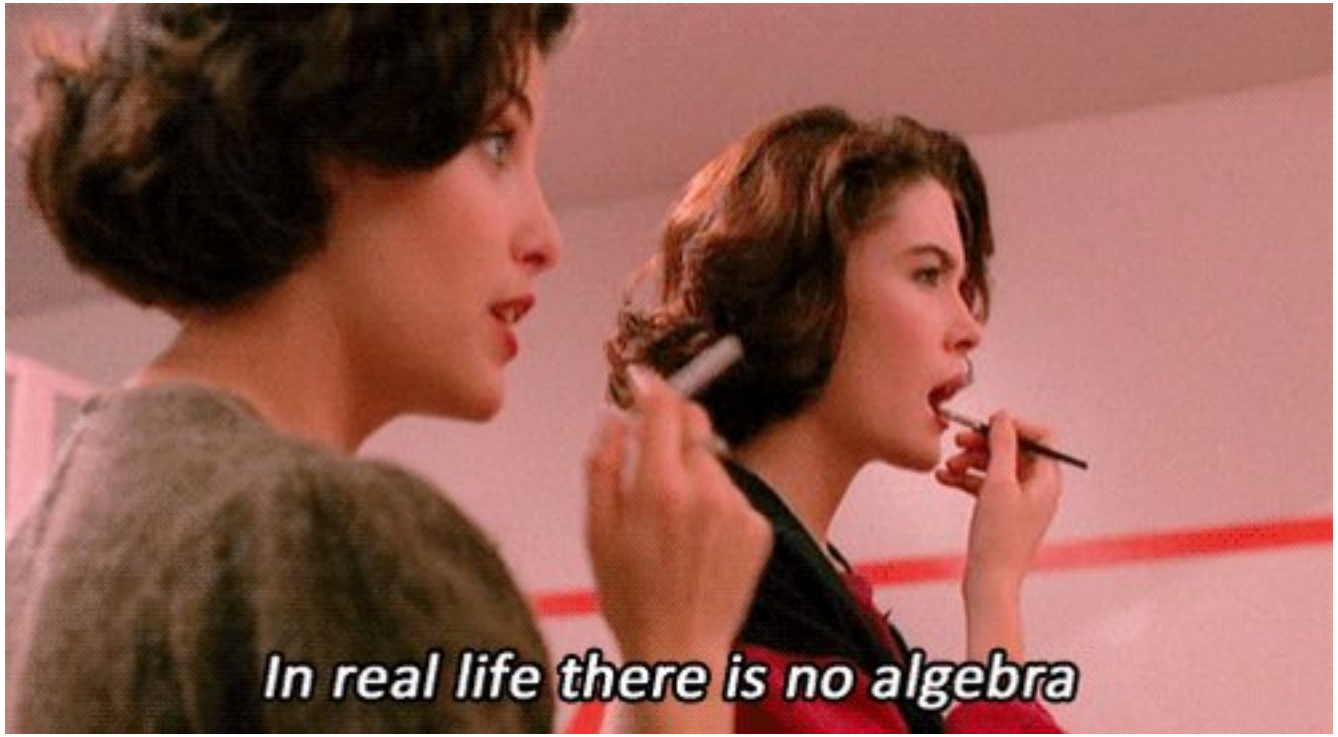
IN GENERAL
- ALGEBRAIC CONSEQUENCES ... PART III

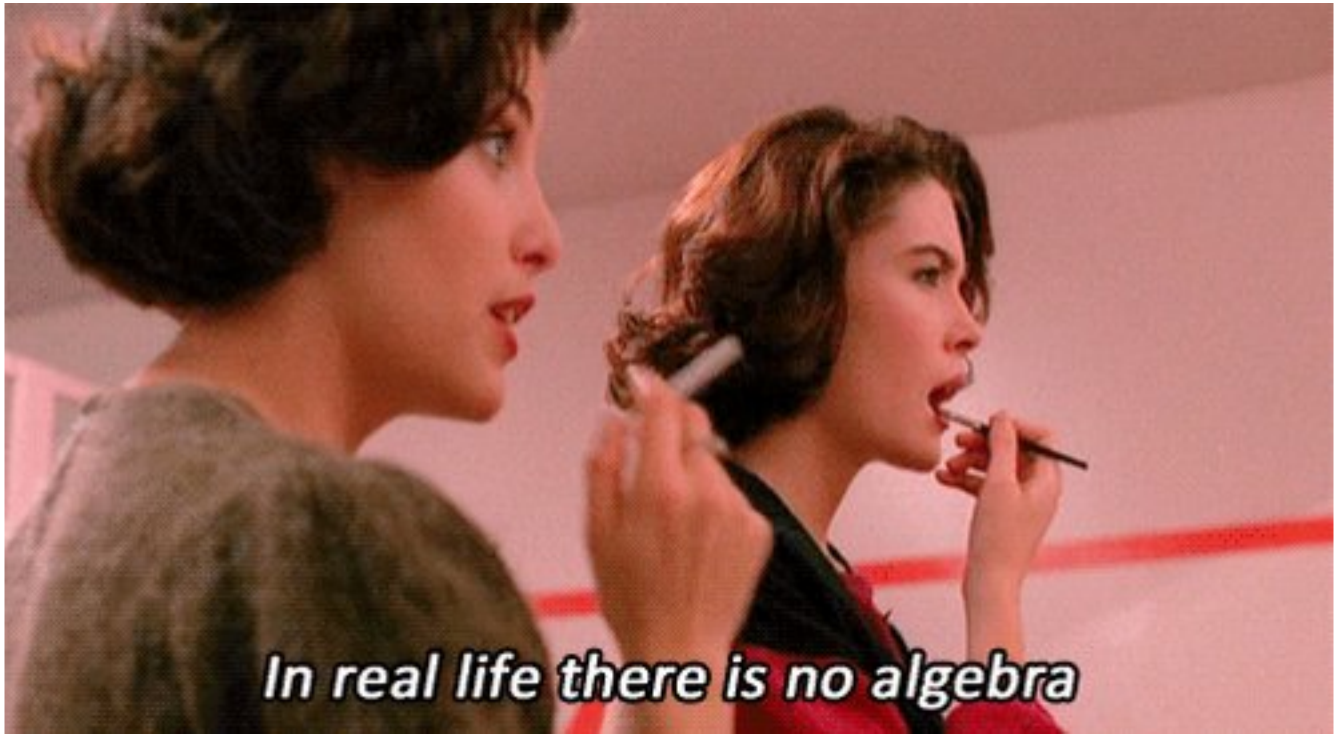
(WHY CAN'T IT ALWAYS BE THAT EASY?)


In real life there is no algebra

PART III

LOOP CONDITIONS

- STRUCTURE → ALGEBRA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THEOREM (HELL + NEŠETŘIL '91): E $\overbrace{\text{FINITE NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\text{Pol(E) TAYLOR}}^{(A)}$

$\Rightarrow$ E HAS LOOP

THEOREM (HELL + NEŠETŘIL '91): $\mathbb{E}$ $\overbrace{\text{FINITE NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\text{Pol}(\mathbb{E})\ \text{TAYLOR}}^{(A)}$

$\Longrightarrow$ $\mathbb{E}$ HAS LOOP

COROLLARY (SIGGERS '11) $\underset{\sim}{A}$ FINITE ALGEBRA, HAS TAYLOR TERM OPERATION

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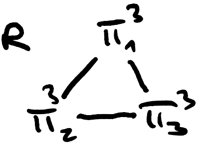
PROOF $\mathcal{E} := \text{Clo}\,\underline{A}$ ... TERM OPERATIONS OF $\underline{A}$

$\mathcal{E}^{(n)}$ ... n-ARY PART OF $\mathcal{E}$

$$\underline{A} \curvearrowright \mathcal{E}^{(3)} : \quad (\underset{\substack{\uparrow \\ \text{OPERATION} \\ \text{OF } \underline{A}}}{f}, (g_1, \underset{\substack{\uparrow \\ \text{3-ARY} \\ \in \mathcal{E}}}{\ldots}, g_6)) \mapsto \underset{\substack{\uparrow \\ \text{3-ARY} \\ \in \mathcal{E}}}{f(g_1 \ldots g_6)}$$

$\Rightarrow$ $\mathcal{E}^{(3)} \leq \underline{A}^{A^3}$

CONSIDER



$\leq \mathcal{E}^{(3)} \times \mathcal{E}^{(3)}$

$\mathbb{E} := \langle \quad \ldots \quad \rangle \leq \mathcal{E}^{(3)} \times \mathcal{E}^{(3)}$ HAS LOOP $\Rightarrow \exists s \in \mathcal{E}^{(6)}$

$$\begin{array}{l} s(\overline{\pi}_1\ \overline{\pi}_2\ \overline{\pi}_1\ \overline{\pi}_3\ \overline{\pi}_2\ \overline{\pi}_3) = \\ \quad\ \ \mid \quad \mid \quad \mid \quad \mid \quad \mid \quad \mid \\ s(\overline{\pi}_2\ \overline{\pi}_1\ \overline{\pi}_3\ \overline{\pi}_1\ \overline{\pi}_3\ \overline{\pi}_2) \end{array}$$

THEOREM (HELL + NEŠETŘIL '91): E $\overbrace{\text{FINITE NON-BIPARTITE GRAPH}}^{(S)}$, $\overbrace{\text{Pol}(E) \text{ TAYLOR}}^{(A)}$

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PROOF $\mathcal{C} := \text{Clo}\,\underline{A}$ ... TERM OPERATIONS OF $\underline{A}$

$\mathcal{C}^{(n)}$ ... n-ARY PART OF $\mathcal{C}$

$\underline{A} \curvearrowright \mathcal{C}^{(3)}$: $(f, (g_1, \dots, g_k)) \mapsto f(g_1 \dots g_k)$

$\underset{\text{OPERATION OF } \underline{A}}{\uparrow}$ $\underset{\text{3-ARY} \in \mathcal{C}}{\uparrow}$ $\underset{\text{3-ARY} \in \mathcal{C}}{\uparrow}$

$\Rightarrow$ $\mathcal{C}^{(3)} \leq \underline{A}^{A^3}$

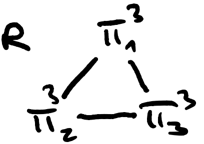
"ALTERNATIVE"

FREE ALGEBRA $F_{\underline{A}}\{x,y,z\}$

$E := \left\langle \left\{ \binom{x}{y}, \binom{y}{x}, \binom{x}{z}, \binom{z}{x}, \binom{x}{y} \binom{y}{z} \right\} \right\rangle$

HAS LOOP

CONSIDER



$\leq \mathcal{C}^{(3)} \times \mathcal{C}^{(3)}$

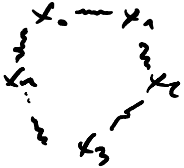
$E := \langle \quad \dots \quad \rangle \leq \mathcal{C}^{(3)} \times \mathcal{C}^{(3)}$ HAS LOOP $\Rightarrow \exists s \in \mathcal{C}^{(6)}$

$$\begin{matrix} s(\overline{\pi_1} & \overline{\pi_2} & \overline{\pi_1} & \overline{\pi_3} & \overline{\pi_2} & \overline{\pi_3}) = \\ | & | & | & | & | & | \\ s(\pi_2 & \overline{\pi_1} & \overline{\pi_3} & \overline{\pi_1} & \overline{\pi_3} & \overline{\pi_2}) \end{matrix}$$

THEOREM (BARTO, KOZIK, NIVEN '07)

NOW: (S)

$$E \text{ FINITE } \overbrace{\underbrace{\text{SMOOTH}}_{\substack{\text{NO SOURCES}\\ \text{NO SINKS}}} \text{ DIGRAPH, } \underbrace{\text{ALGEBRAIC LENGTH } 1}_{\exists x_0 \cdots x_n :}}^{(S)}, \ \overbrace{\text{Pol}(E) \text{ TAYLOR}}^{(A)} \Rightarrow E \text{ LOOP}$$



$\sim \ \in \{\rightarrow, \leftarrow\}$

$(\# \rightarrow) - (\# \leftarrow) = 1$

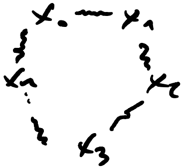
THEOREM (BARTO + KOZIK + NIVEN '07)

NOW: (S)

E FINITE $\overbrace{\underbrace{\text{SMOOTH}}\text{ DIGRAPH, }\underbrace{\text{ALGEBRAIC LENGTH 1}}}^{(S)}$, $\overbrace{\text{Pol(E) TAYLOR}}^{(A)}$ ⇒ E LOOP

NO SOURCES
NO SINKS

$\exists x_0 \cdots x_n$:



$\sim\ \in \{\rightarrow, \leftarrow\}$

$(\# \rightarrow) - (\# \leftarrow) = 1$

COROLLARY (SIGGERS '11) $\underline{A}$ FINITE ALGEBRA, HAS TAYLOR TERM OPERATION
⇒ HAS TERM OPERATION $s(a,r,e,a) = s(r,a,r,e)$

THEOREM (BARTO + KOZIK + NIVEN '07)

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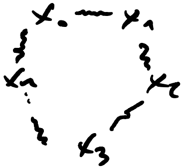


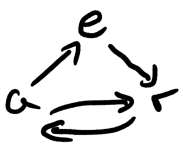
$\rightsquigarrow \in \{\to, \leftarrow\}$

$(\# \to) - (\# \leftarrow) = 1$

COROLLARY (SIGGERS '11) $\underline{A}$ FINITE ALGEBRA, HAS TAYLOR TERM OPERATION

$\Rightarrow$ HAS TERM OPERATION $s(a,r,e,a) = s(r,a,r,e)$

PROOF AS BEFORE:



SMOOTH, ALGEBRAIC LENGTH 1

$\langle \dots \rangle$ IN $F_{\underline{A}}\{a,r,e\} \Rightarrow$ LOOP $\Rightarrow s$

∎

THEOREM (MAROTI + MCKENZIE '07)

$$\overbrace{A \text{ FINITE, } R \subseteq A^n \text{ SYMMETRIC, } n > |A| \text{ PRIME}}^{(S)} , \overbrace{\mathrm{Pol}(R) \text{ TAYLOR}}^{(A)}$$

$$\Longrightarrow R \text{ LOOP}$$

THEOREM (MAROTI + MCKENZIE '07)

$$\overbrace{A \text{ FINITE, } R \subseteq A^n \text{ SYMMETRIC , } n > |A| \text{ PRIME}}^{(S)} \text{ , } \overbrace{\mathrm{Pol}(R) \text{ TAYLOR}}^{(A)}$$

$\Rightarrow$ R LOOP

COROLLARY $\underset{\sim}{A}$ FINITE ALGEBRA , HAS TAYLOR TERM OPERATION

$\Rightarrow$ HAS WEAK NEAR-UNANIMITY OPERATION w :

$$\forall x,y \quad w(x \cdots xy) = w(x \cdots xyx) = \cdots = w(yx \cdots x)$$

THEOREM (MAROTI + MCKENZIE '07)

$$\overbrace{\mathbb{A} \text{ FINITE}, \ R \subseteq A^n \text{ SYMMETRIC}, \ n > |A| \text{ PRIME}}^{(S)}, \ \overbrace{\mathrm{Pol}(R) \text{ TAYLOR}}^{(A)}$$

$\Longrightarrow$ R LOOP

COROLLARY $\underline{A}$ FINITE ALGEBRA, HAS TAYLOR TERM OPERATION

$\Longrightarrow$ HAS WEAK NEAR-UNANIMITY OPERATION w:

$$\forall x,y \quad w(x \cdots xy) = w(x \cdots xyx) = \cdots = w(yx \cdots x)$$

PROOF CONSIDER $F_{\underline{A}}\{x,y\}$, $n > |A|^{A^2}$ PRIME

$$R := \left\langle \left\{ \begin{pmatrix} x \\ \vdots \\ x \\ y \end{pmatrix} \cdots \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix} \right\} \right\rangle \quad n\text{-ARY ON } F_{\underline{A}}\{x,y\}$$

$$R \text{ SYMMETRIC} \Longrightarrow \text{LOOP} \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} \in R$$

$$\Longrightarrow \exists w \in \mathrm{Clo}\,\underline{A} \quad w\left(\begin{pmatrix} x \\ \vdots \\ x \\ y \end{pmatrix}, \ldots, \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix} \right) = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} \quad \square$$

THEOREM (MAROTI + MCKENZIE '07)

$\overbrace{A \text{ FINITE, } R \subseteq A^n \text{ SYMMETRIC}, \; n > |A| \text{ PRIME}}^{(S)}, \; \overbrace{\mathrm{Pol}(R) \text{ TAYLOR}}^{(A)}$

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PROOF CONSIDER $F_{\underline{A}}\{x,y\}$, $n > |A|^{A^2}$ PRIME

$$R := \left\langle \left\{ \begin{pmatrix} x \\ \vdots \\ x \\ y \end{pmatrix} \cdots \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix} \right\} \right\rangle$$

n-ARY ON $F_{\underline{A}}\{x,y\}$

R SYMMETRIC $\Rightarrow$ LOOP $\begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} \in R$

$$\Rightarrow \exists w \in \mathrm{Clo}\,\underline{A} \quad w\left(\begin{pmatrix} x \\ \vdots \\ x \\ y \end{pmatrix}, \ldots, \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix} \right) = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} \qquad \square$$

CYCLIC $R := \left\langle \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ x_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \right\} \right\rangle$ CYCLIC $\Rightarrow$ LOOP

$\square$

GENERAL PATTERN

$V \ldots$ VARIABLE SET , $n \geq 1$

$$R = \{ \underbrace{\begin{pmatrix} r^1_1 \\ \vdots \\ r^1_n \end{pmatrix}, \ldots, \begin{pmatrix} r^m_1 \\ \vdots \\ r^m_n \end{pmatrix}}_{m} \} \subseteq V^n$$

$\Rightarrow$ R DEFINES IDENTITIES Σ_R FOR AN m-ARY FUNCTION f :

$$f(r^1_1 \ldots r^m_1) =$$

$$\vdots$$

$$= f(r^1_n \ldots r^m_n)$$

EXAMPLE

$$R = \{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} z \\ x \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} z \\ y \end{pmatrix} \}$$

$\Rightarrow$ SIGGERS

GENERAL PATTERN

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$\Rightarrow$ SIGGERS

PROPOSITION $\underline{A}$ ALGEBRA $\Rightarrow$ $\exists f \in \mathrm{Clo}\,\underline{A}$ SATISFYING Σ_R

$\Longleftrightarrow$

$\forall \underline{B} \in V(\underline{A}) \ \forall S \subseteq B^n$

$(R \xrightarrow{\text{HOM}} S \Rightarrow S$ HAS LOOP$)$

GENERAL PATTERN

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PROOF "$\Rightarrow$" EASY: APPLY f ON IMAGE OF R IN S

"$\Leftarrow$" IN $F_{\underline{A}}(V)$, CONSIDER $\langle R \rangle$

IDENTITIES FROM LOOP CONDITIONS

VS.

GENERAL IDENTITIES

(STRONG MAL'CEV CONDITIONS)

IDENTITIES FROM LOOP CONDITIONS

VS.

GENERAL IDENTITIES

(STRONG MAL'CEV CONDITIONS)

- ONLY ONE FUNCTION SYMBOL f

  VS.

$$
\begin{aligned}
f_1(\quad) &= g_1(\quad) \\
&\vdots \\
f_n(\quad) &= g_n(\quad)
\end{aligned}
$$

- HEIGHT 1 (NO NESTING $f(g(f \dots)$

- IDENTITIES CONNECTED: $f(\quad) = f(\quad) = f(\quad) \dots$

  VS.

$$
\begin{aligned}
f(\quad) &= f(\quad) \\
&\vdots \\
f(\quad) &= f(\quad)
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$$

IDENTITIES FROM LOOP CONDITIONS
VS.
GENERAL IDENTITIES
(STRONG MAL'CEV CONDITIONS)

- ONLY **ONE FUNCTION SYMBOL** f

  vs.

  $$f_1(\quad) = g_1(\quad)$$
  $$\vdots$$
  $$f_n(\quad) = g_m(\quad)$$

- **HEIGHT 1** (NO NESTING $f(g(f \dots)$)

- IDENTITIES **CONNECTED**: $f(\quad) = f(\quad) = f(\quad) \dots$

  vs.

  $$f(\quad) = f(\quad)$$
  $$\vdots$$
  $$f(\quad) = f(\quad)$$

TAYLOR

$$t(x ? \dots ?) = t(y ? \dots ?)$$
$$\vdots$$
$$t(? \quad ? \, x) = t(? \dots ? \, y)$$

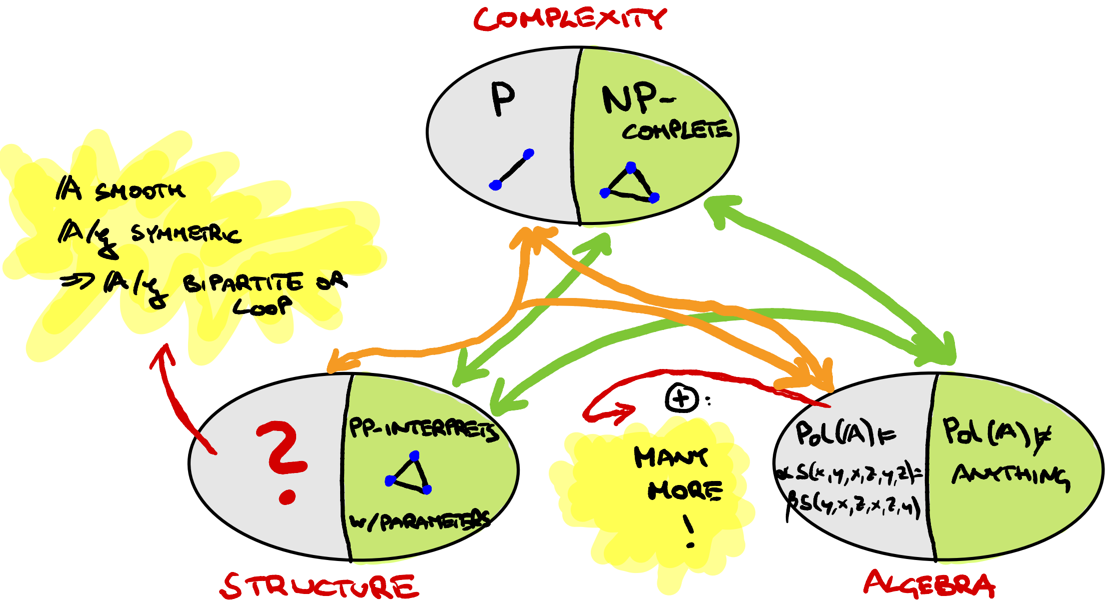
EQUIVALENT TO LOOP COND.

- FOR FINITE $\underline{A}$ (SIGGERS '11)
- FOR IDEMPOTENT $\underline{A}$ (OLŠÁK 18)

CONJECTURE (BODIRSKY + P. '11)

$\mathbb{A}$ INFINITE BUT $\exists \mathcal{G} \le \mathrm{Aut}(\mathbb{A})$:
- $\forall n$ $\mathbb{A}^n/\mathcal{G}$ FINITE ("ORBITS")
- ORBITS HAVE EFFECTIVE DESCRIPTION

Thank you!



HO!

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THERE WERE A COUPLE OF LIES IN THIS TALK.

BONUS TRACK

TAYLOR ⟷ LOOP CONDITION IN IDEMPOTENT ALGEBRAS

(PROOF OF GILLIBERT + JONUŠAS + P.)

LEMMA $\forall n\ \exists$ LOOPCONDITION IDENTITIES Σ_n :

$\forall \underline{A}$ IDEMPOTENT ($\underline{A}$ HAS n-ARY TAYLOR $\Leftrightarrow$ $\mathrm{Clo}\,\underline{A} \models \Sigma_n$)

PROOF LET $R_n := \{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \{x_1 \dots x_{2n}\}^3 \mid$ NOT ALL EQUAL$\}$

$\Sigma_n := \Sigma_{R_n}$ NON-TRIVIAL (NOT ALL EQUAL)

LET $t \in \mathrm{Clo}\,\underline{A}$ TAYLOR, n-ARY

$\Rightarrow t$ SATISFIES IDENTITIES

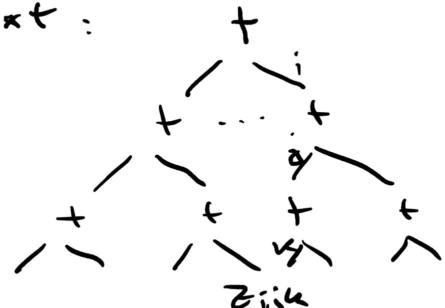
$$t(x_{11} \dots x_{1n}) = t(y_{11} \dots y_{1n})$$

$$\vdots \qquad\qquad\qquad \vdots$$

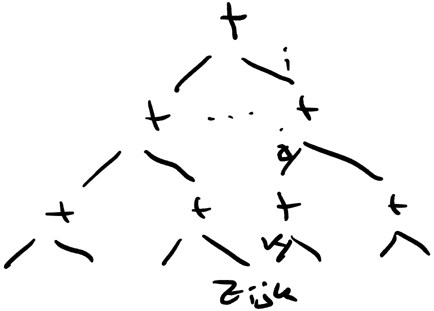
$$t(x_{n1} \dots x_{nn}) = t(y_{n1} \dots y_{nn})$$

- VARIABLES IN DISTINCT ROWS DISJOINT
- $\forall i \quad x_{ii} \neq y_{ii}$

SET $h := t * t * t$:



$h =$



$\in \operatorname{Clo} \underline{A}$

SET $z^a_{ijk} := x_{ij}$, $z^b_{ijk} := y_{ij}$, $z^c_{ijk} = x_{jk}$

$$\left.\begin{aligned} &\left.\begin{aligned} \Rightarrow\ h(z^a_{ijk}) &= t*t(x_{ij}) && \text{(IDEMPOTENCY)}\\ h(z^b_{ijk}) &= t*t(y_{ij}) && ″ \end{aligned}\right\}= \\ &h(z^c_{ijk}) = t*t(x_{jk}) \qquad\qquad ″ \end{aligned}\right\}=$$

$\Rightarrow \operatorname{Clo} \underline{A}$ SATISFIES Σ_{R_n}

LEMMA $n \geq 4 \Rightarrow \Sigma_{R_{n+1}} \rightarrow \Sigma_{R_n}$

COROLLARY $\underline{A}$ IDEMPOTENT TAYLOR

$\Rightarrow \operatorname{Clo} \underline{A} \models \Sigma_{R_4}$

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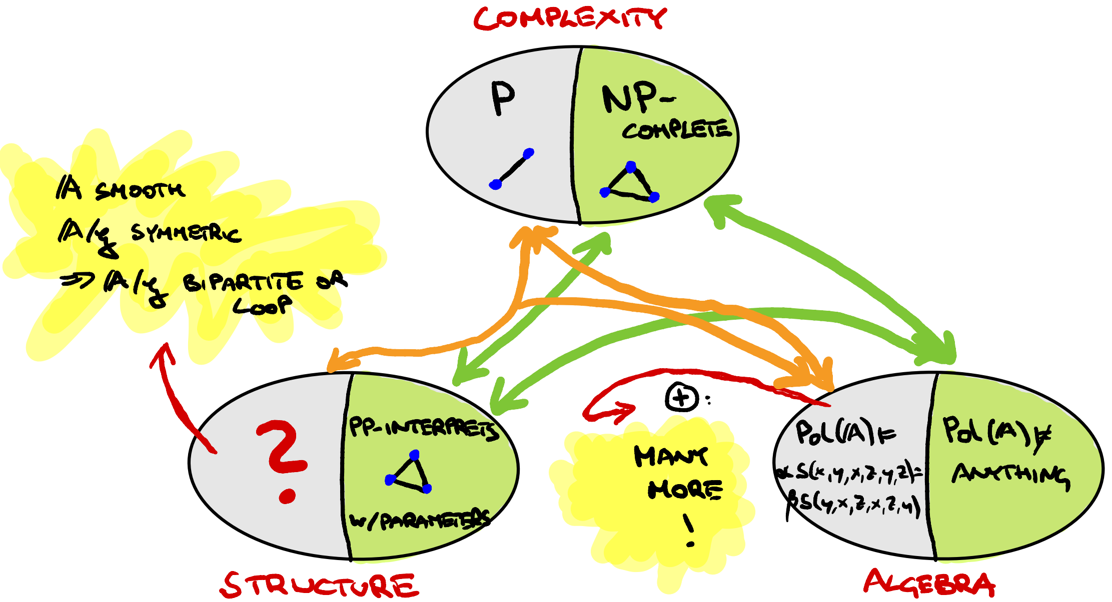

NO!

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