Algebraic Methods for the Complexity of Constraint Satisfaction Problems

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Given variables $x_1, \ldots, x_n$ and some "constraints", decide whether the constraints can be satisfied.

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8				6				3
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7				2				6
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$$x_1 = x_2 + x_3 + x_4$$
$$x_2 = x_3 - x_1$$

- satisfiability of formulas

$$(x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_4 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4)$$

CSP templates

Definition

A *CSP-template* is any relational structure, i.e. a set A together with some r_i-ary relations $R_i \subseteq A^{r_i}$ on A.

$$\mathbb{A} = (A; R_1, R_2, \dots)$$

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Computational problem CSP($\mathbb{A}$):
INPUT: a sentence of the form

$$\exists x_1 \exists x_2 \dots \exists x_n : R_{i_1}(\text{some variables}) \wedge R_{i_2}(\text{more variables}) \wedge \dots \wedge R_{i_k}(\text{maybe the same variables})$$

QUESTION: is the sentence true in $\mathbb{A}$?

Meta-questions

Question 1

Given $\mathbb{A}$, is there a fast (polynomial time) algorithm that solves $\mathrm{CSP}(\mathbb{A})$?

2-SAT vs. 3-SAT

$$\mathbb{A} = \Big(\{0,1\}; (x \vee y), (x \vee \neg y), (\neg x \vee y), (\neg x \vee \neg y)\Big)$$

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$$\mathbb{B} = \Big(\{0,1\}; (x \vee y \vee z), (x \vee y \vee \neg z), \dots, (\neg x \vee \neg y \vee \neg z)\Big)$$

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3-SAT

CSP($\mathbb{B}$) is NP-complete, i.e. (assuming P$\neq$NP) there is no algorithm solving 3-SAT in polynomial time.

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Question 1

Given $\mathbb{A}$, is there a fast (polynomial time) algorithm that solves CSP($\mathbb{A}$)?

Question 2

Can we decide this based on some algebraic invariant of $\mathbb{A}$?

Polymorphisms

Definition

A *homomorphism* between two relational structures (A, R_i) and (B, S_i) is a map $f : A \to B$ such that

$$f_*(R_i) \subseteq S_i$$

A *polymorphism* of arity n of $\mathbb{A}$ is a homomorphism

$$\mathbb{A}^n \to \mathbb{A}$$

Polymorphisms of 2-SAT and 3-SAT

2-SAT has an interesting polymorphism:

$$\begin{aligned} \mathrm{maj} : \{0,1\}^3 \to \{0,1\},\ (x,x,y) &\mapsto x \\ (x,y,x) &\mapsto x \\ (y,x,x) &\mapsto x \end{aligned}$$

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3-SAT does not: every polymorphism is a projection

$$\pi_i : (x_1, \ldots, x_n) \mapsto x_i$$

Dichotomy Theorem

Theorem (Bulatov, Zhuk 2017)

If $\mathbb{A}$ is a **finite** structure, then:

- $\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$, if $\mathbb{A}$ has any "interesting" polymorphism
- if not, then $\mathrm{CSP}(\mathbb{A})$ is NP-complete

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Categorification