Finding dense locally finite subgroups in oligomorphic groups

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TU Dresden

Logic Seminar, Apr 22, 2024

Based on joint work with subsets of Evans, Hubička, Jahel, Nešetřil, and Sabok

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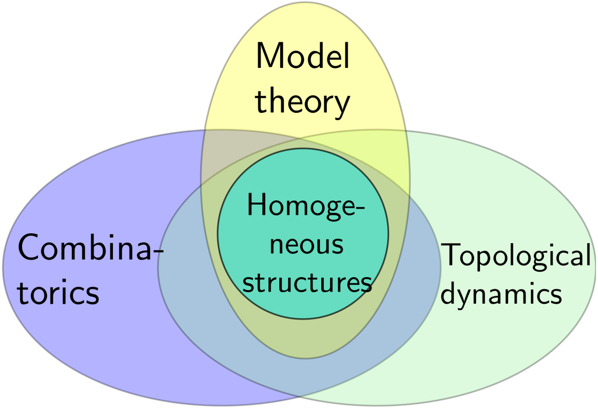
I hope to present some interesting **M**'s.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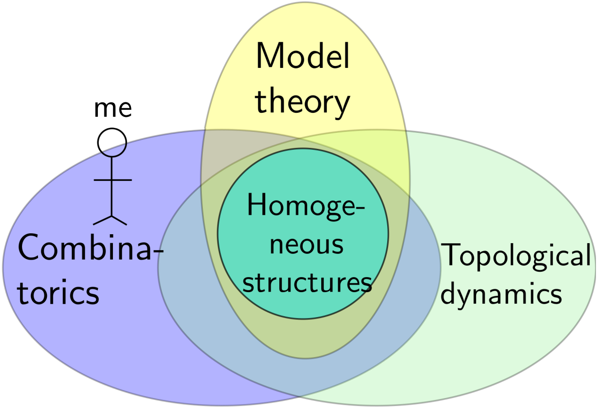
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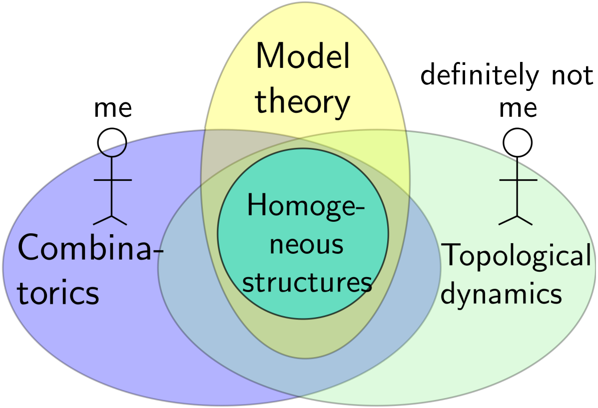
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ω

$\mathrm{Sym}(\omega)$

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Pointwise stabilisers of finite sets form a system of neighbourhoods of the identity.

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In fact, **M** *can be chosen to be homogeneous.*

Homogeneous structures

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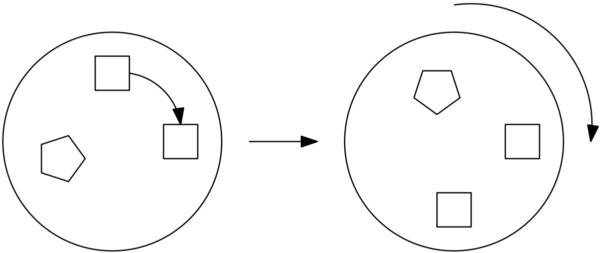
A is homogeneous if every partial automorphism of **A** with finite domain can be extended to an automorphism of **A**.



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Conversely, if $\mathcal{C}$ is a hereditary isomorphism-closed class of finite structures with JEP and AP such that it has only countably many members up to isomorphism then there is a homogeneous structure **M** such that $\mathcal{C} = \mathrm{Age}(\mathbf{M})$. We call this **M** the Fraïssé limit of $\mathcal{C}$ and it is unique up to isomorphism.

Examples

By [Gardiner, 1976] and [Lachlan, Woodrow, 1980], the countable homogeneous graphs are the following:

1. C_5
2. $L(K_{3,3})$
3. Disjoint unions of cliques of the same size (finite or infinite)
4. The countable random graph ($\iff$ all finite graphs)
5. The K_n-free Henson graphs ($\iff$ all finite K_n-free graphs)
6. Complements thereof

Examples II.

- Finite linear orders $\iff$ $(\mathbb{Q}, \leq)$
- Finite sets $\iff$ the countable set
- Finite k-uniform hypergraphs $\iff$ the countable random k-uniform hypergraph
- Finite boolean algebras $\iff$ the countable atomless BA
- Finite tournaments $\iff$ the countable homogeneous tournament
- Finite metric spaces $\iff$ the Urysohn space
- Finite groups $\iff$ Hall's universal locally finite group

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Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

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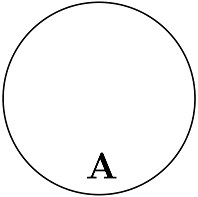

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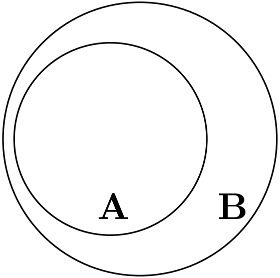

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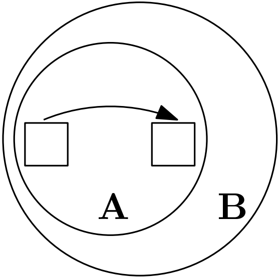

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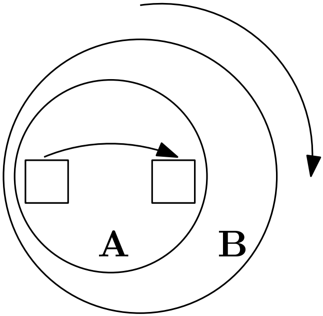

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Theorem (Hrushovski, 1992)

The class of all finite graphs has EPPA.

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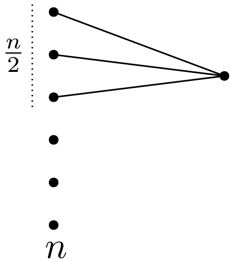

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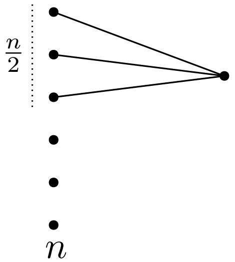

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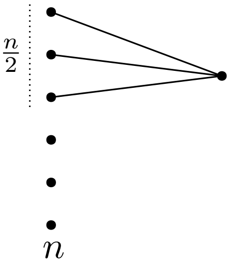

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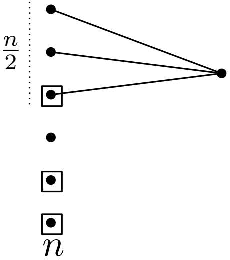

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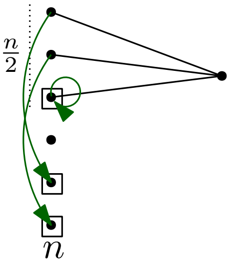

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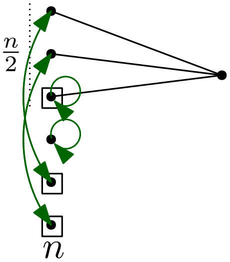

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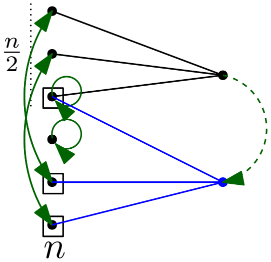

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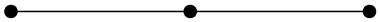

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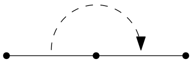

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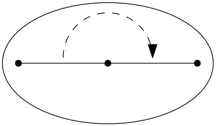

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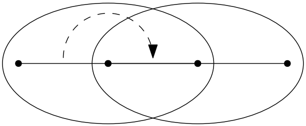

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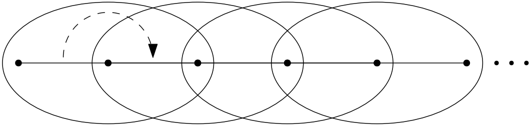

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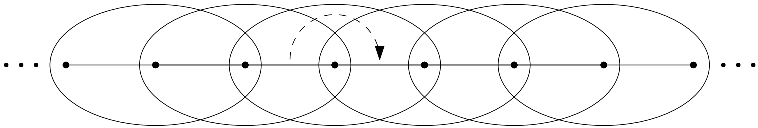

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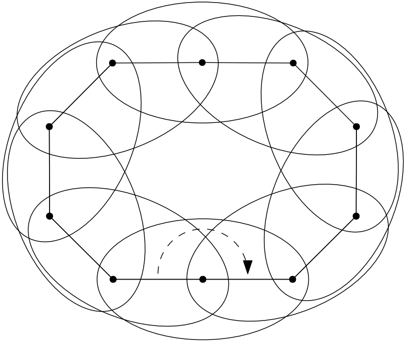

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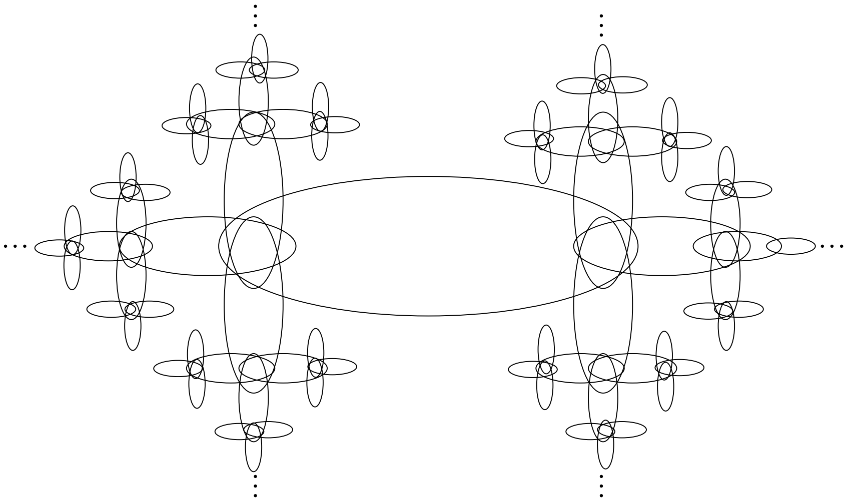

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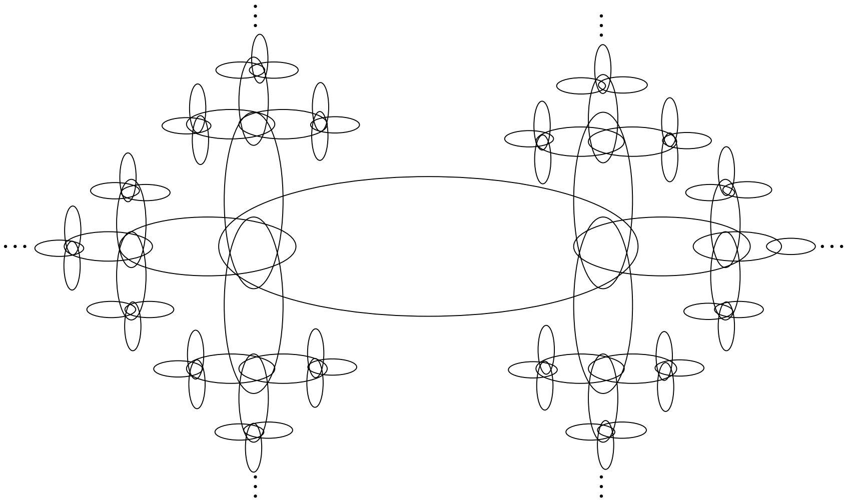

Cf. profinite topology, Hall's theorem, Ribes–Zalesskii theorem, Mackey's construction.

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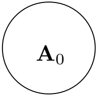

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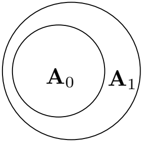

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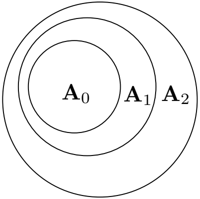

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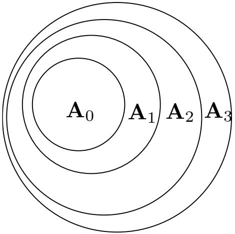

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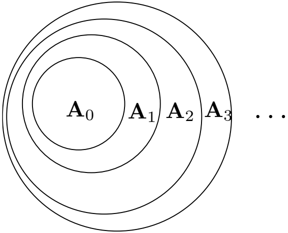

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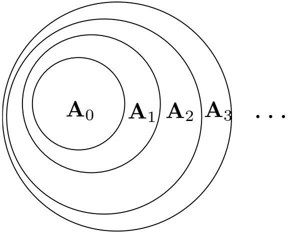

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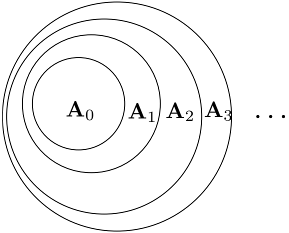

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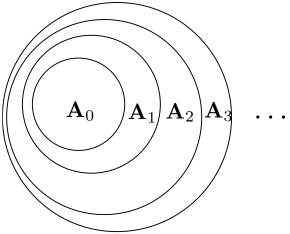

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Theorem (Kechris–Rosendal, 2007)

The class of all substructures of a homogeneous structure **M** *has EPPA if and only if* $\mathrm{Aut}(\mathbf{M})$ *can be written as the closure of a chain of compact subgroups.*

Coherent EPPA

Definition (Coherent EPPA, Solecki'07, Siniora–Solecki'19)

An EPPA-witness **B** for **A** is coherent if there is a map Φ from partial automorphisms of **A** to automorphisms of **B** such that:

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Let **M** *be a homogeneous relational structure. If* $\mathrm{Age}(\mathbf{M})$ *has coherent EPPA then* $\mathrm{Aut}(\mathbf{M})$ *contains a dense locally finite subgroup.*

Coherent EPPA

Definition (Coherent EPPA, Solecki'07, Siniora–Solecki'19)

An EPPA-witness **B** for **A** is coherent if there is a map Φ from partial automorphisms of **A** to automorphisms of **B** such that:

1. $\Phi(f)$ extends f,
2. if $\mathrm{Dom}(g) = \mathrm{Range}(f)$ then $\Phi(gf) = \Phi(g)\Phi(f)$.

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By the way...

If $\mathrm{Aut}(\mathbf{M})$ contains a dense locally finite subgroup then $\mathrm{Age}(\mathbf{M})$ has EPPA. (I think!)

Which classes have EPPA'?

- Graphs [Hrushovski, 1992], K_n-free graphs [Herwig, 1998]
- Relational structures (with forbidden cliques) [Herwig, 2000], [Hodkinson–Otto, 2003]
- Metric spaces [Solecki, 2005; Vershik, 2008], also [Hubička–K–Nešetřil, 2019]
- Two-graphs [Evans–Hubička–K–Nešetřil, 2020]
- Metrically homogeneous graphs [AB-WHKKKP, 2017], [K, 2020]
- Generalised metric spaces [Hubička–K–Nešetřil, 2019+]
- n-partite and semigeneric tournaments [Hubička–Jahel–K–Sabok, 2024]
- Groups [Siniora, 2017]
- ...

Which classes have **coherent** EPPA'?

- Graphs [Bhattacharjee–Macpherson'05]
- K_n-free graphs, relational structures (with forbidden cliques) [Hodkinson–Otto, 2003] + [Siniora–Solecki'19]
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- ~~Two-graphs~~ OPEN
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Every finite set is a coherent EPPA-witness for itself.

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5. Exercise: Verify that $\Phi(gf) = \Phi(g)\Phi(f)$.

□

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Proof (Evans, Hubička, K, Nešetřil, 2021).

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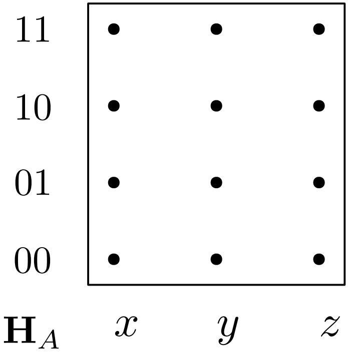

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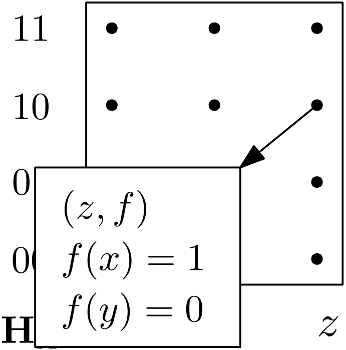

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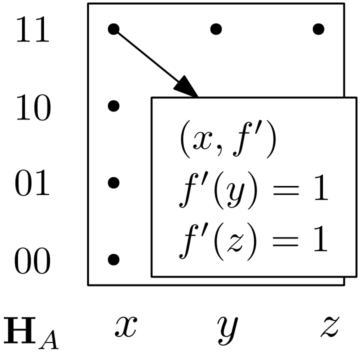

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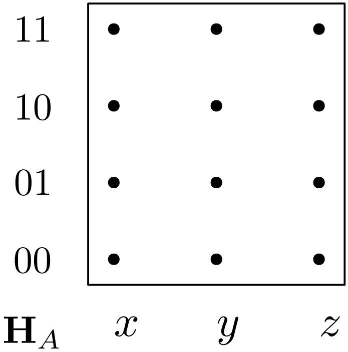

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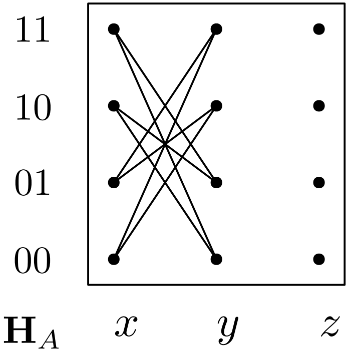

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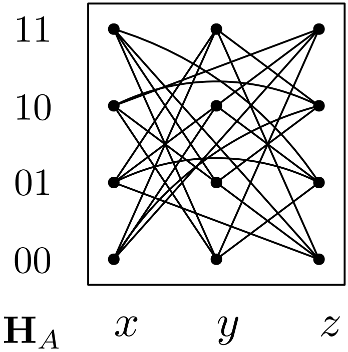

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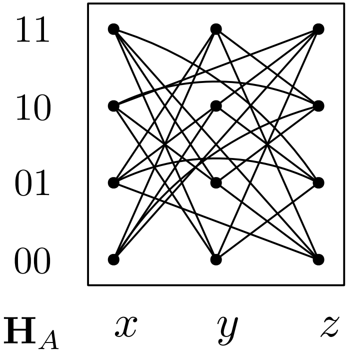

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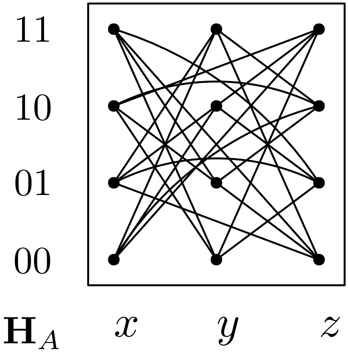

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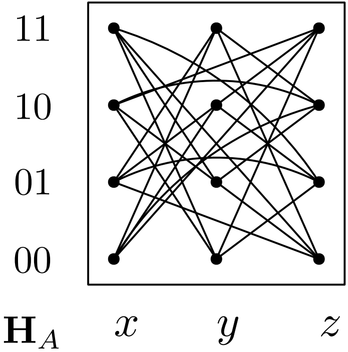

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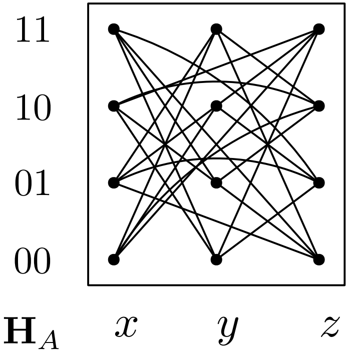

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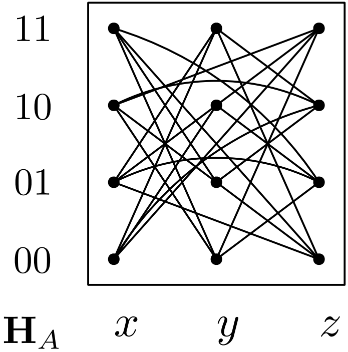

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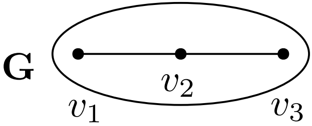

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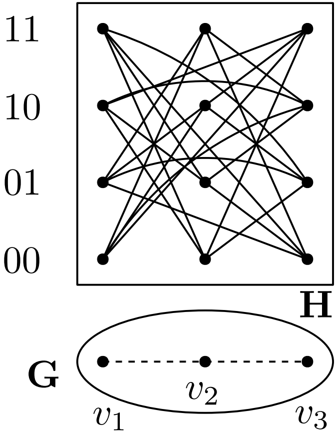

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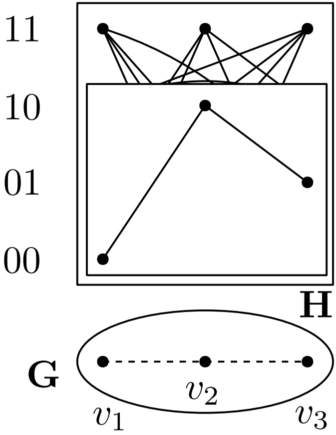

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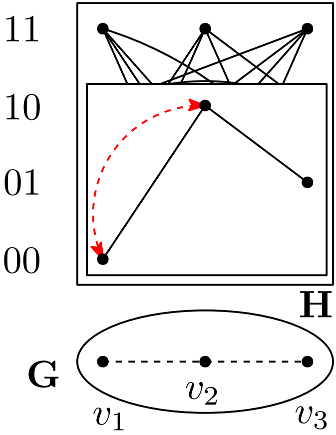

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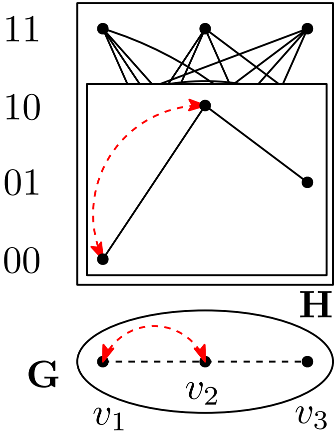

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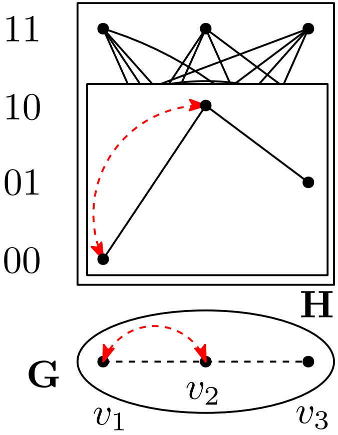

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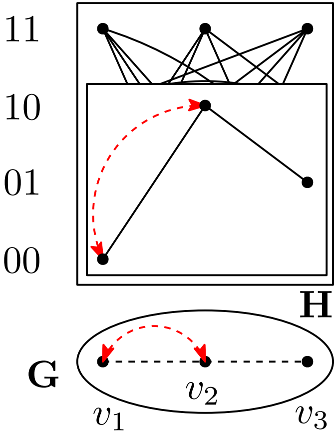

Theorem (Hubička–K–Nešetřil 2022)

Let L be a language where all functions are unary. Given a finite L-structure $\mathbf{A}$ and $n \geq 1$, there is a finite L-structure $\mathbf{B}$ satisfying the following:

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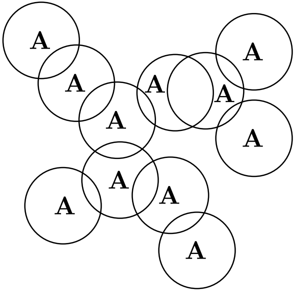

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- Handles almost all known EPPA classes.

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Tournaments	OPEN	OPEN	OPEN

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Is there a class with EPPA but not coherent EPPA? How to disprove coherent EPPA, actually?

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Is coherent EPPA equivalent to some property of $\mathrm{Aut}(\mathbf{M})$?

Exceptions

Class	EPPA	coherent EPPA	DLFS
Groups	[Siniora'17]	[Siniora'17]	YES
Two-graphs	[Evans–Hubička–K–Nešetřil'19]	OPEN	YES
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[Etedadialiabadi–Gao'19]: Ultraextensive spaces..?

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