Minimal operations over permutation groups

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Outline

Constraint Satisfaction Problems

$\tau =$ finite relational language.

Definition 1 ($\mathrm{CSP}(B)$)

Let B be a **fixed** structure.
$\mathrm{CSP}(B)$ is the following computational problem:

- **INPUT:** A **finite** τ-structure A;
- **OUTPUT:** Is there a homomorphism $A \to B$?

- B is finite $\Rightarrow \mathrm{CSP}(B)$ is in NP;
- Many meaningful problems also for infinite B;
- We want to study the **computational complexity** of CSPs.

Examples I

Example 2 (n-colorability for graphs)

Let K_n be the complete graph on n verteces. Then,

- $\mathrm{CSP}(K_n) = n$-colorability problem for graphs;
- NP-complete for $n > 2$ and in P for $n = 2$ (Karp 1972).

Example 3 (NOT ALL EQUAL SAT)

NAE-SAT is the following NP-complete problem (Schaefer 1978):
INPUT: $\mathcal{P}$, a finite set of propositions. $\mathcal{C}$, a finite set of disjunctions of triplets from $\mathcal{P}$;
OUTPUT: Is there an assignment of $\{$TRUE, FALSE$\}$ to the propositions in $\mathcal{P}$ so that each clause in $\mathcal{C}$ has at least one true and one false proposition?
NAE-SAT=CSP(B) for $B := (\{0,1\}; \mathrm{NAE})$ for
$\mathrm{NAE} := \{0,1\}^3 \setminus \{(1,1,1),(0,0,0)\}$.

Examples II

Example 4 (digraph acyclicity)

Consider $(\mathbb{Q}, <)$. Then,

- $\mathrm{CSP}(\mathbb{Q}, <)$ = digraph acyclicity, i.e.
  INPUT: a finite directed graph D;
  OUTPUT Does D contain a finite directed cycle? This problem is in P (indeed, can be solved in linear time) (Kahn 1962).

Example 5 (Solving arithmetic equations)

Consider the $\mathrm{CSP}(B)$ for

$$B := (\mathbb{Z}; \{0\}, \{1\}, \{(x, y, z) | x + y = z\}, \{(x, y, z) | xy = z\}).$$

Then, $\mathrm{CSP}(B)$ is deciding whether a given finite set of arithmetic equations has a solution. Undecidable (Matijasevič 1977).

Polymorphisms I

For B finite, the complexity of $\mathrm{CSP}(B)$ can be captured by the **polymorphisms** of B:

Definition 6 (Polymorphism)

Let $f : B^n \to B$.
f is a **polymorphism** if it **preserves all relations** of B:

$$\begin{pmatrix} a_1^1 \\ \vdots \\ a_k^1 \end{pmatrix}, \ldots, \begin{pmatrix} a_1^n \\ \vdots \\ a_k^n \end{pmatrix} \in R^B \Rightarrow \begin{pmatrix} f(a_1^1, \ldots, a_1^n) \\ \vdots \\ f(a_k^1, \ldots, a_k^n) \end{pmatrix} \in R^B.$$

We call $\mathbf{Pol}(\boldsymbol{B})$ the set of polymorphisms of B.
The **polymorphism clone of** B.

Polymorphisms II

- Unary polymorphism = homomorphism;
- **Projections** to one coordinate **are always polymorphisms**: say that $\pi_i : B^n \to B$ is a projection to the ith coordinate. Then, for

$$\begin{pmatrix} a_1^1 \\ \vdots \\ a_k^1 \end{pmatrix}, \ldots, \begin{pmatrix} a_1^n \\ \vdots \\ a_k^n \end{pmatrix} \in R^B, \begin{pmatrix} \pi_i(a_1^1, \ldots, a_1^n) \\ \vdots \\ \pi_i(a_k^1, \ldots, a_k^n) \end{pmatrix} = \begin{pmatrix} a_1^i \\ \vdots \\ a_k^i \end{pmatrix} \in R^B.$$

- Generalisation of automorphism groups.

Identities I

We are interested in identities satisfied by operations on B. We write identities which hold universally using $\approx$. For example,

$$f(x,y) \approx f(y,x)$$

denotes that f is commutative.

Definition 7 ($h1$-identity)

A **height-one identity** is an identity with exactly one function symbol on each side

Example:

$$f(x,y,y) \approx g(y,x,x)$$

Non-examples:

$$f(x,x,x) \approx x, \qquad f(x,g(x,y),z) \approx g(x,y).$$

Identities II

We say that a set of identities is **trivial** if it is satisfied by projections.
Example:

$$\{f(x,x) \approx f(x,y), \quad f(x,f(y,z)) \approx f(f(x,y),z)\}.$$

Theorem 8 (Siggers 2010+Barto, Opršal, and Pinsker 2018+. . .)

Let B be finite. Then, exactly one of the following holds:

(H) *All sets of $h1$-identites satisfied by polymorphisms in $\mathrm{Pol}(B)$ are trivial;*

(E) $\mathrm{Pol}(B)$ *contains a 6-ary* ***Siggers polymorphism*** $s : B^6 \to B$ *such that*

$$s(x,y,x,z,y,z) \approx s(y,x,z,x,z,y).$$

CSP-dichotomy

- (H) implies that $\mathrm{CSP}(B)$ is NP-hard;
- Bulatov 2017 and Zhuk 2020 proved independently that (E) implies that there is an algorithm in P solving $\mathrm{CSP}(B)$.

Hence, they proved the following, conjectured in (Feder and Vardi 1998):

Theorem 9 (Bulatov 2017; Zhuk 2020)

Let B be a finite relational structure. Then, $\mathrm{CSP}(B)$ is

- *in P if and only if $\mathrm{Pol}(B)$ has a Syggers polymorphism;*
- *NP-complete, otherwise.*

- If P≠NP, there are intermediate problems between P and NP-complete (Ladner 1975);
- We can characterise computational complexity of CSPs in purely algebraic terms.

What about infinite domain CSPs?

Question 1

Are there interesting classes of infinite structures for which the finite domain techniques can be generalised?

Many techniques generalise (with a topological twist) to **ω-categorical** structures.

These countable structures are characterised by their automorphism groups being **oligomorphic**:
for each $n < \omega$, $\mathrm{Aut}(B) \curvearrowright B^n$ has finitely many orbits.

In order for there to be some hope of proving a CSP-dichotomy, we will need to restrict this class a bit more ...

Homogeneous structures

Definition 10 (homogeneous)

A countable structure B is **homogeneous** if any isomorphism between finite substructures of B can be extended to an automorphism.

When a class of finite structures $\mathcal{C}$ forms a **Fraïssé class** we can build a countable homogeneous structure B such that $\mathbf{Age}(\boldsymbol{B})$, i.e. its class of finite substructures, is $\mathcal{C}$. We call B the **Fraïssé limit** of $\mathcal{C}$.

Examples 11

Homogeneous structure	Fraïssé class
Random graph	finite graphs
Generic $\triangle$-free graph	finite $\triangle$-free graphs
$(\mathbb{Q}, <)$	finite linear orders

Finite boundedness

Definition 12 (Finite boundedness)

Homogeneous B is **finitely bounded** if there is a **finite** set $\mathcal{F}$ of τ-structures such that

$$\mathrm{Age}(B) = \mathrm{Forb}^{\mathrm{emb}}(\mathcal{F}).$$

All of the examples in the previous page are finitely bounded.

Definition 13 (Reduct)

Let A and B be structures with the same domain.
A is a **reduct** of B if all relations of A are first-order definable in B.

An example

Example 14 (Monochromatic triangle-free colouring, Burr 1976)

Graphs with edges coloured in red and blue not containing any monochromatic triangle are a Fraïssé class.

So there is an associated (finitely bounded) **universal homogeneous {monochromatic triangle}-free 2-coloured graph** C.

Let D be the reduct of C obtained by "forgetting the colours of the edges".

$\mathrm{CSP}(D)$ is the NP-complete problem:
INPUT: A finite graph G;
OUTPUT: Can the edges of G be coloured so that there is no monochromatic triangle?

The infinite-domain dichotomy conjecture

CSPs for (finite language) reducts of finitely bounded homogeneous structures are in the class NP.

Conjecture 15 (Bodirsky & Pinsker)

Let B be a finite language reduct of a finitely bounded homogeneous structure. Then, $\mathrm{CSP}(B)$ is either in P or NP-complete.

Again, to study $\mathrm{CSP}(B)$, it is important to understand $\mathrm{Pol}(B)$.

Operation clones

Definition 16 (Operation clone)

Let B denote a set. For $n \in \mathbb{N}$, $\mathcal{O}^{(n)}$ denotes the set B^{B^n} of functions $B^n \to B$. We write

$$\mathcal{O} := \bigcup_{n \in \mathbb{N}} \mathcal{O}^{(n)}.$$

An **operation clone** over B is a set $\mathcal{C} \subseteq \mathcal{O}$ such that

- $\mathcal{C}$ contains all projections;
- $\mathcal{C}$ is closed under composition: for $f \in \mathcal{C} \cap \mathcal{O}^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{C} \cap \mathcal{O}^{(m)}$, $f(g_1, \ldots, g_n)$, given by

  $$(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)),$$

  is in $\mathcal{C} \cap \mathcal{O}^{(m)}$.

Topology

We equip $\mathcal{O}^{(n)}$ with the product topology and $\mathcal{O}$ with the sum topology, where B was endowed with the discrete topology.

Given $\mathcal{S} \subseteq \mathcal{O}$, $\langle \mathcal{S} \rangle$ denotes the smallest operation clone containing $\mathcal{S}$. Meanwhile, $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ in $\mathcal{O}$ with respect to the topology we described.

For $\mathcal{S} \subseteq \mathcal{O}$, let $\mathbf{Inv(\mathit{S})}$ be the structure on B whose relations are exactly the relations on B invariant under all $f \in S$. We have that

$$\overline{\langle \mathcal{S} \rangle} = \mathrm{Pol}(\mathrm{Inv}(\mathcal{S})),$$

Minimal clones (and operations)

Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed subclones of $\mathcal{O}$.

Definition 17 (Minimal clone)

We say that $\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.

Definition 18 (almost minimal)

The k-ary operation $f \in \mathcal{D} \setminus \mathcal{C}$ is **almost minimal above** $\mathcal{C}$ if for each $r < k$,

$$\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}.$$

Definition 19 (Minimal operation)

The k-ary operation $f \in \mathcal{D} \setminus \mathcal{C}$ is **minimal above** $\mathcal{C}$ if it is almost minimal and for all $h \in \overline{\langle \mathcal{C} \cup \{f\} \rangle} \setminus \mathcal{C}, f \in \overline{\langle \mathcal{C} \cup \{h\} \rangle}$.

Relations between concepts

Fact 20

There is a correspondence between closed clones which are minimal above $\mathcal{C}$ and closed clones of the form $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ for f minimal.

- When B is finite or ω-categorical in a finite language, for any closed $\mathcal{C} \supsetneq \overline{\langle \mathrm{Aut}(B) \rangle}$, there is $\mathcal{D}$ minimal such that

$$\overline{\langle \mathrm{Aut}(B) \rangle} \subsetneq \mathcal{D} \subseteq \mathcal{C};$$

- For any $G \curvearrowright B$ and $\mathcal{C} \supsetneq \overline{\langle G \rangle}$, there are almost minimal operations above $\overline{\langle G \rangle}$ in $\mathcal{C}$.

Why care about minimal polymorphisms?

Definition 21 (Essentially unary and essential operations)

f is **essentially unary** if there is unary g and $1 \leq i \leq k$ such that

$$f(x_1, \ldots, x_k) \approx g(x_i).$$

Otherwise, f is **essential**.

- To prove that $\mathrm{CSP}(B)$ is in P, we often need to find some essential operation of low arity.
- $\mathrm{Pol}(B)$ will contain minimal operations above $\overline{\langle \mathrm{Aut}(B) \rangle}$;
- We prove that for B ω-categorical with $\mathrm{CSP}(B)$ in P we can always find a binary essential operation!

Some terminology for operations

We define some operations in virtue of the identities they satisfy:

- **Ternary quasi-majority**:

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x);$$

- **Quasi-Malcev:**

$$M(x,y,y) \approx M(y,y,x) \approx M(x,x,x);$$

- A **quasi-semiprojection** is a k-ary f such that there is an $i \in \{1,\ldots,k\}$ and a unary operation g such that whenever **at least two of the a_j equal each other,**

$$f(a_1,\ldots,a_k) = g(a_i).$$

Rosenberg's five types theorem

The following generalises Rosenberg's five types theorem (Rosenberg 1986) for idempotent algebras (i.e. the case of $G = \{1\}$):

Theorem 22 (Five types theorem, Bodirsky and Chen 2007)

Let $G \curvearrowright B$. Let f be minimal above $\overline{\langle G \rangle}$. Then, up to permuting its variables, f is of one of the following five types:

1. *a unary function;*
2. *a binary function;*
3. *a ternary quasi-majority operation;*
4. *a quasi-Malcev operation;*
5. *a k-ary quasi-semiprojection for some $k \geq 3$.*

Some improvements in the oligomorphic case

Theorem 23 (Four types, oligomorphic case, Bodirsky and Chen 2007; Bodirsky 2021)

Let $G \curvearrowright B$ be an oligomorphic permutation group on a countably infinite B. Let f be minimal above $\overline{\langle G \rangle}$. Then, f is of one of the following four types:

1. *a unary function;*
2. *a binary function;*
3. *a ternary quasi-majority operation;*
4. *a k-ary quasi-semiprojection for some $3 \leq k \leq 2r - s$, where r is the number of G-orbitals and s is the number of G-orbits.*

- No quasi-Malcev;
- Upper bound on arity of quasi-semiprojections.

Our results I

Theorem 24 (Three Types Theorem, Marimon and Pinsker 2024)

Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Let f be almost minimal above $\overline{\langle G \rangle}$. Then, f is of one of:

1. *a unary function;*
2. *a binary function;*
3. ~~*a ternary quasi-majority operation;*~~
4. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

f is an **orbit-semiprojection** if there is an $i \in \{1, \ldots, k\}$ and a unary operation $g \in \overline{G}$ such that **whenever at least two of the a_j lie in the same orbit,**

$$f(a_1, \ldots, a_k) = g(a_i).$$

Our results II

- We classify **almost minimal** operations. Recall

$$\text{minimal} \Rightarrow \text{almost minimal};$$

- This is a strict improvement on Bodirsky and Chen 2007: **Oligomorphic permutation groups never act freely;**

- We can classify almost minimal operations above $\overline{\langle G \rangle}$ for any permutation group. The remaining two cases are
  - G is a Boolean group acting freely on B with $|G| > 2$;
  - $\mathbb{Z}_2$ acting freely on B.

- We will completely specify the possible behaviour of f on orbits;

- We also have more information on the binary operations.

Our results III

Theorem 25 (Boolean case, Marimon and Pinsker 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s-many orbits and $|G| > 2$. Let f be an almost minimal operation above $\langle G \rangle$. Then, f is of one of the following types:

1. *f is unary;*
2. *f is binary;*
3. *f is a ternary twisted minority;*
4. *f is a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

A **twisted minority** is a ternary operation such that for all $\beta \in G$,

$$\mathfrak{m}(y, x, \beta x) \approx \mathfrak{m}(x, \beta x, y) \approx \mathfrak{m}(x, y, \beta x) \approx \mathfrak{m}(\beta y, \beta y, \beta y).$$

Our results IV

Theorem 26 ($\mathbb{Z}_2$ case, Marimon and Pinsker 2024)

Let $\mathbb{Z}_2$ act freely on B with s-many orbits. Let f be an almost minimal operation above $\langle \mathbb{Z}_2 \rangle$. Then, f is of one of the following types:

1. *f is unary;*
2. *f is a ternary twisted minority;*
3. *f is an odd majority;*
4. *f is, up to permuting its variables, an odd Malcev;*
5. *f is a k-ary orbit-semiprojection for $2 \leq k \leq s$.*

An **odd majority** m is a quasi-majority such that for γ the non-identity element in $\mathbb{Z}_2$,

$$m(y, x, \gamma x) \approx m(x, \gamma x, y) \approx m(x, y, \gamma x) \approx m(y, y, y).$$

An **odd Malcev** is a quasi-Malcev such that $M(x, \gamma y, z)$ is an odd majority.

On the existence of these operations

- For $|\mathrm{Orb}(G)| \leq 2$, f can only be unary or binary;
- For $|\mathrm{Orb}(G)| \geq 3$, the non-binary operations in our classifications **always exist as almost minimal** above $\overline{\langle G \rangle}$;
- For $|\mathrm{Orb}(G)| \geq 3$, orbit-semiprojections always exist as minimal above $\overline{\langle G \rangle}$;
- Twisted minorities, odd majorities and odd Malcev operations should frequently not exist as minimal;
- e.g. $|\mathrm{Orb}(G)| = 3 \Rightarrow$ no twisted minority minimal above $\overline{\langle G \rangle}$;

An example of a proof

Lemma 27 (not free $\Rightarrow$ no quasi-Malcev)

Let $G \curvearrowright B$ be such that the action of G on B is not free. Then, no almost minimal function over $\overline{\langle G \rangle}$ can be a quasi-Malcev operation.

Proof.

Take $\alpha \in G, a, b, c \in B$ such that $\alpha(a) = a, \alpha(b) = c$. Suppose $M(x, y, z)$ is almost minimal and quasi-Malcev. Then, $h(x, y) = M(x, \alpha x, y)$ is essentially unary. If it depends on the first argument,

$$M(a, a, a) = h(a, a) = h(a, b) = M(a, a, b) = M(b, b, b) \ ,$$

contradicting injectivity of $M(x, x, x) \in \overline{\langle G \rangle}$. Similarly, if $h(x, y)$ depends on the second argument,

$$M(c, c, c) = M(a, a, c) = h(a, c) = h(b, c) = M(b, c, c) = M(b, b, b) \ ,$$

contradicting injectivity of $M(x, x, x) \in \overline{\langle G \rangle}$. Thus, $h(x, y)$ depends on both arguments, contradicting the almost minimality of M. □

A question of Bodirsky on binary polymorphisms

Definition 28

For B finite or ω-categorical, we say that B is a **model complete core** if $\overline{\langle \mathrm{Aut}(B) \rangle} = \mathrm{End}(B)$.

For CSPs it is sufficient to look at model complete cores.

Finding binary essential polymorphisms is very helpful in building arguments for why a CSP is in P. In particular, in the open problems section of his book on CSPs Bodirsky asks:

Question 2 (Question 24 in Bodirsky 2021)

Does every countably infinite ω-categorical model complete core with an essential polymorphism also have a binary essential polymorphism?

A counterexample

Answer: NO, one can build an ω-categorical model complete core whose polymorphism clone is $\overline{\langle \mathrm{Aut}(B) \cup \{f\} \rangle}$, where B consists of three infinite predicates partitioning a countably infinite set and f is a 3-ary orbit-semiprojection minimal above it.

- One needs $\mathrm{Aut}(B)$ to have at least 3 orbits for a counterexample;
- We actually prove: **whenever $\mathbf{CSP}(B)$ is not NP-hard, $\mathbf{Pol}(B)$ has a binary essential polymorphism.**

Why problems in P lie above binary polymorphisms

Theorem 29 (Marimon and Pinsker 2024)

Suppose that B a finite or ω-categorical model complete core and $\mathrm{Aut}(B) \curvearrowright B$ is not the free action of a Boolean group on B (this is always the case if B is ω-categorical). Suppose that $\mathrm{CSP}(B)$ is not NP-hard. Then, $\mathrm{Pol}(B)$ contains a binary essential polymorphism.

- This result can be phrased in purely universal algebraic terms (not relying on $P \neq NP$);

Proof.

Suppose that $\mathrm{Pol}(B) \cap \mathcal{O}^{(2)} = \overline{\langle \mathrm{Aut}(B) \rangle} \cap \mathcal{O}^{(2)}$. Then, all ternary operations in $\mathcal{C}$ must be almost minimal. So $\mathrm{Pol}(B) \cap \mathcal{O}^{(3)}$ consists entirely of essentially unary operations and orbit-semiprojections. We can then show that these will only satisfy trivial $h1$-identities. From this we can prove that $\mathrm{CSP}(B)$ is NP-complete. □

Some thoughts on the presence of symmetry

- **Our result is very false if one looks at rigid structures:** even on a two-element domain there are CSPs in P for structures whose minimal polymorphism above $\langle 1 \rangle$ is a ternary majority (cf. Schaefer 1978);

- Adding finitely many constants does not change the complexity of a CSP. So people often do that when studying these problems;

- Indeed, many of the early results on finite domain CSPs rely on adding constants for all elements of the structure, making it rigid (this is less needed with modern techniques);

- Hence, if you study finite domain CSPs, these results can be quite surprising.

Problems for the future

- Do minimal twisted minorities, odd majorities and odd-Malcevs ever exist?
- Can we find general conditions on $G \curvearrowright B$ showing these do not exist as minimal above $\overline{\langle G \rangle}$?
- Can we find general conditions on $G \curvearrowright B$ that imply that there is no binary minimal operations above $\overline{\langle G \rangle}$?
- Is there an ω-categorical structure $\mathcal{M}$ such there is no binary minimal operation above $\overline{\langle \mathrm{Aut}(M) \rangle}$?

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An extra on binary operations I

Definition 30 (identity multrigraph)

$\mathcal{C}$ is a set of unary functions on B. The **identity multigraph** $\mathcal{C}^\star$ is the $\mathcal{L}$-structure where $\mathcal{L} = \{R_a | a \in B\}$ with domain $\mathcal{C}$, where for $\alpha, \beta \in \mathcal{C}$,

$$R_a(\alpha, \beta) \text{ if and only if } \alpha a = \beta a.$$

Let $G \curvearrowright B$. A homomorphism of the identity multigraphs $\Gamma : G^\star \to \overline{G}^\star$ is **binary making** if

- $|\mathrm{Im}(\Gamma)| > 1$;
- $\Gamma \neq F_\alpha$ for $\alpha \in \overline{G}$, where $F_\alpha(\beta) = \alpha\beta$.

An extra on binary operations II

Theorem 31 (Marimon and Pinsker 2024)

Let $G \curvearrowright B$. There is a one-to-one correspondence between:

- ***binary making homomorphism*** $\Gamma : G^\star \to \overline{G}^\star$*;*
- *binary f almost minimal above $\overline{\langle G \rangle}$.*

This is given by the map $\Gamma \mapsto f_\Gamma$, where

$$f_\Gamma(x, \beta x) := \Gamma(\beta)(x)$$

The existence of binary making homomorphism is non-trivial.

Question 3 (Something to play with)

Take your favourite permutation group $G \curvearrowright B$. Does it have binary making homomorphisms?