When measures don't care about structure

and when they do

Paolo Marimon Samuel Braunfeld Colin Jahel

TU Wien

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Outline

Homogeneous structures

Definition 1

$\mathcal{M}$ is **homogeneous** if any isomorphism between finite substructures extends to an automorphism of the whole structure.

When a class of finite structures $\mathcal{C}$ forms a **Fraïssé class** we can build a countable homogeneous structure $\mathcal{M}$ whose **age**, i.e. its class of finite substructures, is $\mathcal{C}$. We call $\mathcal{M}$ the **Fraïssé limit** of $\mathcal{C}$.

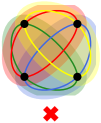
Examples 2

Homogeneous structure	Fraïssé class
Random graph	finite graphs
Generic $\triangle$-free graph	finite $\triangle$-free graphs
Universal homogeneous 3-hypergraph	finite 3-hypergraphs

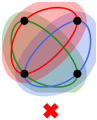
Some examples to keep in mind I

We focus on homogeneous 3-hypergraphs.

- **Universal homogeneous $\mathbf{3}$-hypergraph** $\mathcal{R}_3$;
  A **$\mathbf{3}$-hypergraph** has a ternary hyperedge relation taking distinct triplets of vertices.
- **Generic tetrahedron-free $\mathbf{3}$-hypergraph** $\mathcal{H}$;
  A **tetrahedron** consists of four vertices such that each three of them form a hyperedge.

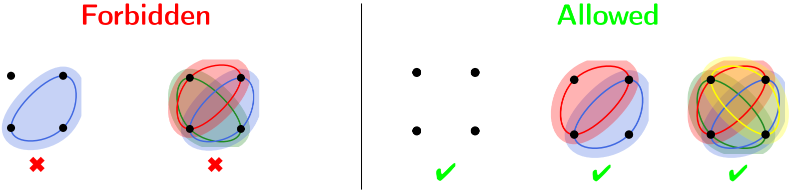


- **Generic $\mathcal{K}_4^-$-free $\mathbf{3}$-hypergraph** $\mathcal{K}$;
  $\mathcal{K}_4^-$ consists of four vertices with three 3-hyperedges.



Some examples to keep in mind II

- **Universal homogeneous two-graph** $\mathcal{G}$;
  A **two-graph** is a 3-hypergraph such that any four vertices have an even number of hyperedges.



This structure is a reduct of the random graph, obtained by drawing a hyperedge whenever three vertices have an odd number of edges.

The previous structures satisfy **free amalgamation**, while $\mathcal{G}$ does not.

Invariant Random Expansions

Fix countable homogeneous $\mathcal{M}$ in a countable relational language $\mathcal{L}$.
Let $\mathcal{L}'$ be a (distinct) countable relational language and $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}'$.

Definition 3

$\mathrm{Struc}_{\mathcal{L}'}(M) =$ expansions of $\mathcal{M}$ by $\mathcal{L}'$-relations.
It has a topology with a basis given by

$$[\![\phi(\overline{a})]\!] = \{N \in \mathrm{Struc}_{\mathcal{L}'}(M) | N \vDash \phi(\overline{a})\},$$

where $\phi(\overline{x})$ is a quantifier-free $\mathcal{L}'$-formula and $\overline{a}$ is a tuple from M of length $|\overline{x}|$.

Definition 4

An **invariant random expansion** (IRE) of $\mathcal{M}$ to $\mathcal{L}'$ is a Borel probability measure on $\mathrm{Struc}_{\mathcal{L}'}(M)$ which is invariant under the action of $\mathrm{Aut}(M)$ on M.

An example: S_∞-invariant measures

IREs of $(\mathbb{N}, =)$ are called S_∞-invariant measures.

Example 5 (The Random graph)

For each pair of vertices from $\mathbb{N}$, toss a coin to decide whether they form an edge or not. This induces an S_∞-invariant measure μ_R on $\mathrm{Struc}_{\{E\}}(\mathbb{N})$ concentrating on the isomorphism type of the random graph.

Example 6 $(\mathbb{Q}, <)$

There is a unique S_∞-invariant measure on the space of linear orderings of a countable set $\mathrm{LO}(\mathbb{N}) \subseteq \mathrm{Struc}_{\{<\}}(\mathbb{N})$. It concentrates on the isomorphism type of $(\mathbb{Q}, <)$. For any $a_1, \ldots, a_n \in \mathbb{N}$ it gives

$$\mu(a_1 < \cdots < a_n) = \frac{1}{n!}.$$

S_∞-invariant measures

We have a nice way of representing S_∞-invariant measures due to a classical theorem in probability (Aldous 1981, Hoover 1979 and Kallenberg 1997). Moreover,

Theorem 7 (Ackerman, Freer, and Patel 2016)

Let $\mathcal{M}$ be a homogeneous relational structure. Then, tfae:

- *there is an S_∞-invariant measure concentrated on the isomorphism type of $\mathcal{M}$;*
- *$\mathcal{M}$ has trivial algebraic closure.*

There are also many natural examples of S_∞-invariant measures which do not concentrate on any S_∞-orbit (Ackerman, Freer, Kruckman, and Patel 2017).

What about IREs on other structures?

We want to understand IREs for other homogeneous structures $\mathcal{M}$.

Example to keep in mind: for a homogeneous 3-hypergraph $\mathcal{M}$ we want to understand the IREs of $\mathcal{M}$ by a binary relation concentrating on the space of graph expansions of $\mathcal{M}, \mathrm{GRAPH}(\mathcal{M})$.

All S_∞-invariant measures on $\mathrm{GRAPH}(\mathbb{N})$ will also give an IRE for $\mathcal{M}$.

But can we get more?

SPOILER: In many cases we cannot.

A motivation: invariant Keisler measures

Definition 8 (Keisler measure)

A **Keisler measure** on $\mathcal{M}$ in the variable x is a finitely additive **probability** measure on $\mathrm{Def}_x(M)$:

- $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for disjoint X and Y;
- $\mu(M) = 1$.

We want to study Keisler measures **invariant** under automorphisms. We call these **invariant Keisler measures** (IKMs):

$$\mu(X) = \mu(\sigma \cdot X) \text{ for } \sigma \in \mathrm{Aut}(M),$$

where $\sigma \cdot \phi(M, \overline{a}) = \phi(M, \sigma(\overline{a}))$.

These correspond to (regular) Borel probability measures on $S_x(M)$ invariant under the natural action of $\mathrm{Aut}(M) \curvearrowright S_x(M)$.

IKMs on ω-categorical structures

What do we know about IKMs on ω-categorical structures?

- It is useful to study **ergodic measures**. These are well-behaved and every measure can be written as an integral average of them:

$$\mu(X) = \int_{\mathrm{Erg}_x(M)} \nu(X)\mathrm{d}\nu;$$

- For $\mathcal{M}$ NIP, the space of IKMs is well-understood (Ensley 1996, BJM 2024);
- For **ergodic** μ and $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$,

$$\mu(\phi(x,a) \wedge \psi(x,b)) = \mu(\phi(x,a))\mu(\psi(x,b)). \qquad (\diamondsuit)$$

This is extremely helpful in understanding the IKMs for ω-categorical binary structures.

Example: measures in homogeneous graphs

Theorem 9 (Measures on the Random graph, Albert 1994)

Let μ be an IKM for the random graph R (in the variable x). Then, there is a unique measure ν on $[0,1]$ such that

$$\mu(\phi(x, A, B)) = \int_0^1 p^{|A|}(1-p)^{|B|}\mathrm{d}\nu,$$

where for finite and disjoint $A, B \subseteq R, \phi(x, A, B)$ asserts that x is connected to all of A and none of B.

Theorem 10 (Measures on the generic $\triangle$-free graph, Albert 1994)

The generic triangle free graph has a unique IKM corresponding to the unique invariant type p asserting that x is disconnected from everything.

What about IKMs on homogeneous ternary structures?

Issue: there is no analogue of ($\diamondsuit$) for measures of more complex intersections such as

$$\mu(\phi(x, ab) \wedge \psi(x, ac) \wedge \xi(x, bc)). \qquad (\heartsuit)$$

This can be seen from results we will give later.

For the homogeneous graphs, $\mu(\phi(x, a) \wedge \psi(x, b))$ does not depend on $\mathrm{tp}(ab)$ for a and b adequately independent. Can we say something similar for $\heartsuit$?

SPOILER: Yes for some homogeneous free amalgamation structures. But the universal homogeneous two-graph suggests that the picture is much more complicated.

How to view a type as an expansion

Let $S'_x(M)$ be the space of non-realised types. Consider a type $p \in S'_x(M)$ for $\mathcal{M}$ a homogeneous 3-hypergraph. We can associate to p a graph expansion $\mathcal{M}^*_p$ of $\mathcal{M}$ where

$$\mathcal{M}^*_p \vDash E(a,b) \text{ if and only if } R(a,b,x) \in p.$$

Definition 11

We say that the the type space $S'_x(M)$ is **representable in** $\mathrm{Struc}_{\mathcal{L}'}(M)$ if there is an injective $\mathrm{Aut}(M)$-map[1]

$$\Gamma : S'_x(M) \to \mathrm{Struc}_{\mathcal{L}'}(M).$$

We say that $S'_x(M)$ is **represented by** $S = \mathrm{Range}(\Gamma)$.

[1] i.e. a continuous map between the two compact topological spaces preserving the action of $\mathrm{Aut}(M)$.

The connection between IKMs and IREs

Let $\mathfrak{M}'_x(M)$ be the space of IKMs for $\mathcal{M}$ in the variable x whose support contains no realised type.

Corollary 12 (Braunfeld, Jahel, and M. 2024)

Let $S'_x(M)$ be representable by S in $\mathrm{Struc}_{\mathcal{L}'}(M)$. Then, there is an isomorphism between $\mathfrak{M}'_x(M)$ and the space of IREs of $\mathcal{M}$ to $\mathcal{L}'$ concentrating on S.

Theorem 13 (Braunfeld, Jahel, and M. 2024)

For every homogeneous $\mathcal{M}$, there is a language $\mathcal{L}^{pr}$ such that $S'_x(M)$ is representable in $\mathrm{Struc}_{\mathcal{L}^{pr}}(M)$.

To each relation in $\mathcal{L}$ we associate various relations of lower arity to code how x behaves with respect to $\mathcal{M}$.

Structure Independence

Definition 14 (Structure independence)

We say that an IRE μ of $\mathcal{M}$ to $\mathcal{L}'$ is **structure independent** if μ is actually S_∞-invariant.

Definition 15 (Age of a measure)

For an IRE μ of $\mathcal{M}$ to $\mathcal{L}'$ let

$$\mathrm{Age}(\mu) := \{\mathbf{H}^* \in \mathrm{Struc}_{\mathcal{L}'}(H) | H \in \mathrm{Age}(M), \mu(\mathbf{H}^*) > 0\}.$$

Question 1

For which $\mathcal{M}$ and $\mathcal{F}'$ do we get that all IREs of $\mathcal{M}$ such that $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{F}'$ are structure independent?

Invariant Random Expansions with n-DAP I

Crane and Towsner 2018 study IREs of structures with **disjoint n-amalgamation** for all n.

Definition 16 (Disjoint n-amalgamation)

Write $[n]$ for $\{1, \ldots, n\}$. For $I \subseteq [n]$, let $\mathcal{F}_I$ denote the set of structures in $\mathcal{F}$ with domain I.
A **disjoint n-amalgamation problem:** for each $I \subseteq \{1, \ldots, n\}$ of size $n-1$, let $A_I \in \mathcal{F}_I$ be such that for $I \neq J$,

$$A_I \upharpoonright (I \cap J) = A_J \upharpoonright (I \cap J).$$

Solution: $A \in \mathcal{F}_{[n]}$ such that for each $I \subseteq \{1, \ldots, n\}$ of size $n-1$,

$$A \upharpoonright I = A_I.$$

So $\mathcal{F}$ has **disjoint n-amalgamation** (n-DAP) if all such problems have a solution.

Invariant Random Expansions with n-DAP II

- The random graph and universal homogeneous 3-hypergraph have n-DAP for all n;
- All other 3-hypergraphs we mentioned do not have 4-DAP;
- Crane and Towsner 2018 give a representation theorem for IREs of structures with n-DAP for all n;
- In particular, if all relations of $\mathcal{L}'$ have **smaller arity than the smallest arity in** $\mathcal{L}$, then all IREs of $\mathcal{M}$ to $\mathcal{L}'$ are **structure independent**;
- By the IKM-IRE correspondence, the space of IKMs on the universal homogeneous k-hypergraph corresponds to the space of S_∞-invariant measures concentrating on the space of $(k-1)$-hypergraphs;
- This answers an open question of Albert 1994 and disproves a conjecture of Ensley 1996 (BJM 2024).

IREs to linear orders

What if instead we look at classes $\mathcal{F}'$ for which it is easy to prove that all IREs with $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{F}'$ are structure independent?

Theorem 17 (Angel, Kechris, and Lyons 2014, rephrased)

Let $\mathcal{M}$ be a homogeneous hypergraph such that $\mathrm{Age}(M)$ has the free amalgamation property. There is a unique IRE μ of $\mathcal{M}$ to $\{<\}$ with $\mathrm{Age}(\mu) \restriction_{\{<\}} \subseteq \mathrm{LO}$, the space of finite linear orders. For $a_1, \ldots, a_n \in M$,

$$\mu(a_1 < \cdots < a_n) = \frac{1}{n!}.$$

- Note: they actually prove structure independence of any μ with $\mathrm{Age}(\mu) \restriction_{\{<\}} \subseteq \mathrm{LO}$!
- See Jahel and Tsankov 2022 for a strong generalisation.

From finite combinatorics to measures

The following Lemma was particularly inspiring to us.
For $\mathbf{H}, \mathbf{G}$ hypergraphs, write $N_{\text{ind}}(\mathbf{H}, \mathbf{G})$ for the number of embeddings of $\mathbf{H}$ in $\mathbf{G}$. Given orderings $<_H$ and $<_G$ on $\mathbf{H}$ and $\mathbf{G}$ respectively, let $N_{\text{ord}}(\mathbf{H}, \mathbf{G})$ denote the number of embeddings respecting the ordering.

Lemma 18 (cf. Lemma 2.1 in Angel, Kechris, and Lyons 2014)

Suppose that for all $\mathbf{H} \in \mathcal{F}_{[k]}$, and $\epsilon > 0$ there is $\mathbf{G} \in \mathcal{F}$ such that $N_{\text{ind}}(\mathbf{H}, \mathbf{G}) > 0$ and for all orderings $<_G$ of $\mathbf{G}$ and $<_H$ of $\mathbf{H}$,

$$\left| \frac{N_{\text{ord}}(\mathbf{H}, \mathbf{G})}{N_{\text{ind}}(\mathbf{H}, \mathbf{G})} - \frac{1}{k!} \right| < \varepsilon.$$

Then, there is a unique IRE μ of $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$ such that $\mathrm{Age}(\mu) \restriction_{\{<\}} \subseteq \mathrm{LO}$, corresponding to the S_∞-invariant one.

Some notation for our results

Our setting: Two Fraïssé classes $\mathcal{F}$ and $\mathcal{F}'$ in languages $\mathcal{L}, \mathcal{L}'$. For $\mathbf{H} \in \mathcal{F}_{[r]}$ and $\mathbf{H}' \in \mathcal{F}'_{[r]}$, write $\mathbf{H} * \mathbf{H}'$ for the **free superposition** of the two structures. Write $\mathcal{F}^* = \mathcal{F} * \mathcal{F}'$ for the class of free superpositions of structures in $\mathcal{F}$ and $\mathcal{F}'$.
We have a homogeneous structure $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$. We study IREs μ of $\mathcal{M}$ to $\mathcal{L}'$ such that $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{F}'$.

Notation 19

Let $\mathbf{H}, \mathbf{G} \in \mathcal{F}$. Let Θ be a family of embeddings of $\mathbf{H}$ in $\mathbf{G}$. Let $\mathbf{H}^*, \mathbf{G}^* \in \mathcal{F}^*$ be such that $\mathbf{H}^*_{\restriction \mathcal{L}} = \mathbf{H}$ and $\mathbf{G}^*_{\restriction \mathcal{L}} = \mathbf{G}$. Then, we write

$$N_\Theta(\mathbf{H}^*, \mathbf{G}^*)$$

for the number of embeddings in Θ that are also embeddings of $\mathbf{H}^*$ in $\mathbf{G}^*$.

From finite combinatorics to structure independence

Lemma 20 (Braunfeld, Jahel, and M. 2024)

Let $\mathbf{H}' \in \mathcal{F}'_{[r]}$ *and* $\mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[r]}$. *Suppose that for all* $\epsilon > 0$ *there is* $n > r$ *and* $\mathbf{G} \in \mathcal{F}_{[n]}$ *with a family* Θ_i *of embeddings of* $\mathbf{H}_i$ *in* $\mathbf{G}$ *such that for all* $\mathbf{G}' \in \mathcal{F}'_{[n]}$,

$$\left| \frac{N_{\Theta_1}(\mathbf{H}_1^*, \mathbf{G}^*)}{|\Theta_1|} - \frac{N_{\Theta_2}(\mathbf{H}_2^*, \mathbf{G}^*)}{|\Theta_2|} \right| < \varepsilon, \qquad (\clubsuit)$$

where $\mathbf{G}^* := \mathbf{G} * \mathbf{G}', \mathbf{H}_i^* := \mathbf{H}_i * \mathbf{H}'$.
Then for any IRE μ *of* $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$ *such that* $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subset \mathcal{F}'$,

$$\mu(\mathbf{H}_1 * \mathbf{H}') = \mu(\mathbf{H}_2 * \mathbf{H}').$$

If we can do this for all $\mathbf{H}' \in \mathcal{F}'_{[r]}, \mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[r]}$, we get structure independence!

A strategy for structure independence

- **What we need:** Given $\mathbf{H}' \in \mathcal{F}'_{[r]}, \mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[r]}$, we want to find a large $\mathbf{G} \in \mathcal{F}$ and a family of embeddings Θ_i of $\mathbf{H}_i$ in $\mathbf{G}$ such that no matter how we add a structure from $\mathcal{F}'$ onto $\mathbf{G}$, there is still a similar proportion of copies of $\mathbf{H}_1 * \mathbf{H}'$ and $\mathbf{H}_2 * \mathbf{H}'$ (counting among the embeddings Θ_i);
- **Method:**
  - Find a large uniform r-hypergraph $\mathbb{K}$. We will need $\mathbb{K}$ to be dense enough. Moreover, it must be so that we can glue copies of $\mathbf{H}_1$ and $\mathbf{H}_2$ onto the hyperedges of $\mathbb{K}$ and still get a structure in $\mathcal{F}$;
  - Build $\mathbf{G}$ by gluing independently at random copies of $\mathbf{H}_1$ and $\mathbf{H}_2$ on the hyperedges of $\mathbb{K}$;
  - $\Theta_i =$ embeddings of $\mathbf{H}_i$ in an r-hyperedge of $\mathbb{K}$;
  - If the growth rate of structures in $\mathcal{F}'$ is slow enough, $\mathbf{G}$ will satisfy (♣).

k-overlap closedness

Definition 21 (k-overlap closed class)

Let $k <$ the minimal arity in $\mathcal{L}$. $\mathcal{F}$ is ***k*-overlap closed** if for all $r \geq k$ there is $N \geq r$ such that for all $n \geq N$ there is r-uniform hypergraph on n vertices satisfying the following conditions:

1. There are at least $C(r)n^{k+\epsilon(r)}$ many hyperedges;
2. No two hyperedges intersect in more than k points;
3. If structures from $\mathcal{F}_{[r]}$ are pasted into the hyperedges, the resulting structure is in $\mathcal{F}$.

We prove that having k-overlap closed age is sufficient to get structure independence for IREs to a k-ary language;

Some k-overlap closed classes

With probabilistic methods and some combinatorics we prove:

- Tetrahedron-free 3-hypergraphs are 2-overlap closed;
- Any k-transitive homogeneous structure in a $k+1$-ary language where all omitted substructures are $(k+1)$-irreducible has k-overlap closed age. By Conant 2017, these are **simple**;
- In particular, all ternary simple 2-transitive homogeneous structures with free amalgamation have 2-overlap closed age. From Koponen 2018, we know there are $2^{\aleph_0}$ such structures;
- Any free amalgamation class of arity $(k+1)$ is 1-overlap closed. It is unclear whether they are also all k-overlap closed;
- Any 2-transitive finitely bounded free amalgamation homogeneous structure in a ternary language has 2-overlap closed age (e.g. the universal homogeneous $\mathcal{K}_4^-$-free 3-hypergraph).

The main Theorem

Theorem 22 (Braunfeld, Jahel, and M. 2024)

Let $\mathcal{F}$ be a k-overlap closed Fraïssé class and let $\mathcal{C}$ be a hereditary class of $\mathcal{L}'$-structures with labelled growth rate $o(e^{n^{k+\delta}})$ for every $\delta > 0$. Then any IRE μ of $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$ such that $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{C}$ is structure independent.

- In the proof we build a large enough $\mathbf{G} \in \mathcal{F}_{[n]}$ on n vertices whose probability of satisfying $\clubsuit$ tends to 1 as $n \to \infty$;
- For example, we get structure independence for the IREs by graphs of the following 3-hypergraphs:
  - The generic tetrahedron-free 3-hypergraph;
  - the universal homogeneous 3-hypergraph;
  - the generic $\mathcal{K}_4^-$-free 3-hypergraph.
- In general, we proved k-overlap closedness for very large classes of structures, and this theorem applies to them.

An application to Keisler measures

Recently, people have been interested in comparing the following two sets of formulas (which capture a notion of 'smallness'):

- $F(\emptyset) :=$ formulas forking over $\emptyset$;
- $\mathcal{O}(\emptyset) :=$ formulas which are assigned measure zero by every IKM.

- $F(\emptyset) \subseteq \mathcal{O}(\emptyset)$ in any theory;
- $F(\emptyset) = \mathcal{O}(\emptyset)$ in stable theories;
- In Chernikov, Hrushovski, Kruckman, Krupinski, Moconja, Pillay, and Ramsey 2021, they give the first examples of simple theories where $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$;
- I found the first simple ω-categorical examples of $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ (Marimon 2023). However, these were not one-based;
- Some original motivation for studying structure independence is that if we could prove it for the generic tetrahedron-free 3-hypergraph, we would have a one-based simple example where $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$.

A dichotomy for Keisler measures

Corollary 23 (Braunfeld, Jahel, and M. 2024)

Let $\mathcal{M}$ be a simple homogeneous structure in a finite ternary language whose age has free amalgamation and such that $\mathrm{Aut}(M)$ is 2-transitive. Then, any invariant Keisler measure for $\mathcal{M}$ in x is structure independent. Moreover,

1. *EITHER: $\mathrm{Age}(M)$ has disjoint n-amalgamation for all n. In this case there is an IKM μ assigning positive measure to every non-forking formula;*
2. *OR: $\mathrm{Age}(M)$ fails disjoint n-amalgamation for some n. In this case,*
$$F(\emptyset) \subsetneq \mathcal{O}(\emptyset).$$

When DO measures care about structure?

Our results also offer an heuristic for when we might expect there to be structure dependent IREs for expansions to lower arity languages (and so structure dependent IKMs).

In particular, if $\mathrm{Age}(M)$ has slow labelled growth compared to the arity of its language, we expect any similar strategy to fail.

For example, the class of two-graphs has slower growth than the class of graphs (and there is no way for it to be k-overlap closed either).

The two-graph

Theorem 24 (M. 2023, cf. Jahel 2021)

The universal homogeneous two-graph $\mathcal{G}$ has a unique invariant Keisler measure μ in the singleton variable. For $\phi(x, a_1, \ldots, a_n)$ isolating $\mathrm{tp}(d/a_1, \ldots, a_n)$, we have that

$$\mu(\phi(x, a_1, \ldots, a_n)) = \left(\frac{1}{2}\right)^{n-1}.$$

- The proof exploits that, fixing a vertex, we can find a copy of the random graph canonically embedded in $\mathcal{G}$;
- A similar uniqueness result holds for the higher arity versions of the two-graph (i.e. the kay-graphs).

Problems we are contemplating

- **What can we say about IREs of $\mathcal{M}$ to languages of larger arity?**
- Structure independence stops making sense.
  But there is an analogue in Crane and Towsner 2018 for which there is some hope;
- **Can we show that the IREs of any free amalgamation class to a language of lower arity are structure independent?**
- It is unclear whether our techniques can be adapted to free-amalgamation classes with very sparse omitted substructures.
  But hopefully better combinatorial arguments work;
- **Can we say more about when and how structure independence fails?**
- Here I am interested in IKMs for ternary reducts of binary homogeneous structures.

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