When measures don't care about structure

and when they do

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Outline

Homogeneous structures

We work with structures in a finite relational language which are **homogeneous**: any isomorphism between finite substructures extends to an automorphism of the whole structure.

When a class of finite structures $\mathcal{C}$ forms a **Fraïssé class** we can build a countable homogeneous structure $\mathcal{M}$ whose **age**, i.e. its class of finite substructures, is $\mathcal{C}$. We call $\mathcal{M}$ the **Fraïssé limit** of $\mathcal{C}$.

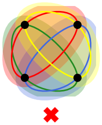
Examples 1

Homogeneous structure	Fraïssé class
Random graph	finite graphs
Generic $\triangle$-free graph	finite $\triangle$-free graphs
Universal homogeneous 3-hypergraph	finite 3-hypergraphs

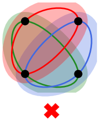
Some examples to keep in mind I

We consider homogeneous structures in a finite relational language. Usually, we will take all relations in the language to be of the same arity $k+1$ and ask that $\mathrm{Aut}(M)$ acts k-transitively on M.
Our main focus later will be homogeneous 3-hypergraphs.

- **Universal homogeneous $\mathbf{3}$-hypergraph** $\mathcal{R}_3$;
  A **$\mathbf{3}$-hypergraph** has a ternary hyperedge relation taking distinct triplets of vertices.
- **Generic tetrahedron-free $\mathbf{3}$-hypergraph** $\mathcal{H}$;
  A **tetrahedron** consists of four vertices such that each three of them form a hyperedge.

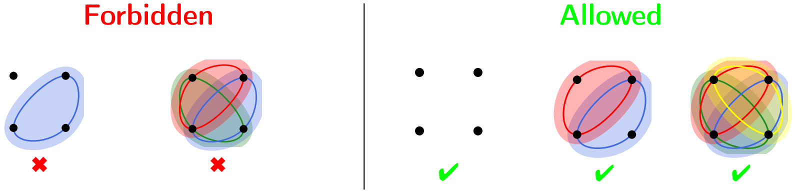


- **Generic $\mathcal{K}_4^-$-free $\mathbf{3}$-hypergraph** $\mathcal{K}$;
  $\mathcal{K}_4^-$ consists of four vertices with three 3-hyperedges.



Some examples to keep in mind II

- **Universal homogeneous two-graph $\mathcal{G}$;**
  A **two-graph** is a 3-hypergraph such that any four vertices have an even number of hyperedges.



This structure is a reduct of the random graph, obtained by drawing a hyperedge whenever three vertices have an odd number of edges.

Invariant Random Expansions

Fix a homogeneous structure $\mathcal{M}$ in a countable relational language $\mathcal{L}$. Let $\mathcal{L}'$ be a (distinct) countable relational language and $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}'$.

Definition 2

$\mathrm{Struc}_{\mathcal{L}'}(M)$ is the space of expansions of $\mathcal{M}$ by $\mathcal{L}'$-relations. It has a topology with a basis of clopen sets given by

$$[\![\phi(\overline{a})]\!] = \{N \in \mathrm{Struc}_{\mathcal{L}'}(M) | N \vDash \phi(\overline{a})\},$$

where $\phi(\overline{x})$ is a quantifier-free $\mathcal{L}'$-formula and $\overline{a}$ is a tuple from M of length $|\overline{x}|$.

Definition 3

An **invariant random expansion** (IRE) of $\mathcal{M}$ to $\mathcal{L}'$ is a Borel probability measure on $\mathrm{Struc}_{\mathcal{L}'}(M)$ which is invariant under the action of $\mathrm{Aut}(M)$ on M.

An example: S_∞-invariant measures

IREs of $(\mathbb{N}, =)$ are called S_∞-invariant measures.

Example 4 (The Random graph)

For each pair of vertices from $\mathbb{N}$, toss a coin to decide whether they form an edge or not. This induces an S_∞-invariant measure μ_R on $\mathrm{Struc}_{\{E\}}(\mathbb{N})$ concentrating on the isomorphism type of the random graph.

Example 5 $(\mathbb{Q}, <)$

There is a unique S_∞-invariant measure on the space of linear orderings of a countable set $\mathrm{LO}(\mathbb{N}) \subseteq \mathrm{Struc}_{\{<\}}(\mathbb{N})$. It concentrates on the isomorphism type of $(\mathbb{Q}, <)$. For any $a_1, \dots, a_n \in \mathbb{N}$ it gives

$$\mu(a_1 < \cdots < a_n) = \frac{1}{n!}.$$

Some facts on S_∞-invariant measures

- Every S_∞-invariant measure has a representation according to a classical theorem of probability (Aldous 1981; Hoover 1979; Kallenberg 1997);
- We know from Ackerman, Freer, and Patel 2016 that for homogeneous $\mathcal{M}$ there is an S_∞-invariant measure concentrating on the isomorphism type of $\mathcal{M}$ if and only if $\mathcal{M}$ has trivial algebraic closure;
- In the case above, either $\mathcal{M}$ is interdefinable with one of the five reducts of $\mathbb{Q}$ (and there is a unique measure concentrating on $\mathcal{M}$) OR there are continuum many such measures (Ackerman, Freer, Kwiatkowska, and Patel 2017);
- There are also many natural examples of S_∞-invariant measures which do not concentrate on any S_∞-orbit (Ackerman, Freer, Kruckman, and Patel 2017).

What about IREs on other structures?

We want to understand IREs for other homogeneous structures $\mathcal{M}$.

An important case to keep in mind: for a homogeneous 3-hypergraph $\mathcal{M}$ we want to understand the IREs of $\mathcal{M}$ by a binary relation concentrating on the space of graph expansions of $\mathcal{M}, \mathrm{GRAPH}(\mathcal{M})$.

Clearly, all S_∞-invariant measures on $\mathrm{GRAPH}(\mathbb{N})$ will also give an IRE for $\mathcal{M}$: in the latter case we are only asking invariance under fewer permutations than in the S_∞-invariant case!

But can we get more? We will see that in many cases we cannot.

A motivation: invariant Keisler measures

Definition 6 (Keisler measure)

A **Keisler measure** on $\mathcal{M}$ in the variable x is a finitely additive **probability** measure on $\mathrm{Def}_x(M)$:

- $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for disjoint X and Y;
- $\mu(M) = 1$.

We want to study Keisler measures **invariant** under automorphisms. We call these **invariant Keisler measures** (IKMs):

$$\mu(X) = \mu(\sigma \cdot X) \text{ for } \sigma \in \mathrm{Aut}(M),$$

where $\sigma \cdot \phi(M, \overline{a}) = \phi(M, \sigma(\overline{a}))$.

These correspond to (regular) Borel probability measures on $S_x(M)$ invariant under the natural action of $\mathrm{Aut}(M) \curvearrowright S_x(M)$.

IKMs on ω-categorical structures

What do we know about IKMs on ω-categorical structures?

- It is useful to study **ergodic measures**. These are well-behaved and every measure can be written as an integral average of them:

$$\mu(X) = \int_{\mathrm{Erg}_x(M)} \nu(X) \mathrm{d}\nu;$$

- For $\mathcal{M}$ NIP, the space of Keisler measures is well-understood (Ensley 1996, BJM 2024);
- For **ergodic** μ and $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$,

$$\mu(\phi(x,a) \wedge \psi(x,b)) = \mu(\phi(x,a))\mu(\psi(x,b)). \qquad (\diamondsuit)$$

  This is extremely helpful in understanding the IKMs for ω-categorical binary structures.

Example: measures in homogeneous graphs

Theorem 7 (Measures on the Random graph, Albert 1990)

Let μ be an IKM for the random graph R (in the variable x). Then, there is a unique measure ν on $[0,1]$ such that

$$\mu(\phi(x,A,B)) = \int_0^1 p^{|A|}(1-p)^{|B|}\mathrm{d}\nu,$$

where for finite and disjoint $A, B \subseteq R, \phi(x,A,B)$ asserts that x is connected to all of A and none of B.

Theorem 8 (Measures on the generic △-free graph, Albert 1990)

The generic triangle free graph has a unique IKM corresponding to the unique invariant type p asserting that x is disconnected from everything.

What about IKMs on homogeneous ternary structures?

Issue: there is no analogue of ($\diamondsuit$) for measures of more complex intersections such as

$$\mu(\phi(x, ab) \wedge \psi(x, ac) \wedge \xi(x, bc)). \qquad (\heartsuit)$$

This can be seen from results we will give later.

For the homogeneous graphs, $\mu(\phi(x, a) \wedge \psi(x, b))$ does not depend on $\mathrm{tp}(ab)$ for a and b adequately independent. Can we say something similar for $\heartsuit$?

SPOILER: Yes for some homogeneous free amalgamation classes of 3-hypergraphs, but the universal homogeneous two-graph suggests that the picture is much more complicated.

How to view a type as an expansion

Let $S'_x(M)$ be the space of non-realised types. Consider a type $p \in S'_x(M)$ for $\mathcal{M}$ a homogeneous 3-hypergraph. We can associate to p a graph expansion $\mathcal{M}^*_p$ of $\mathcal{M}$ where

$$\mathcal{M}^*_p \vDash E(a,b) \text{ if and only if } R(a,b,x) \in p.$$

Something similar can be done for an arbitrary relational structure (with a suitable choice of language).

Definition 9

We say that the the type space $S'_x(M)$ is **representable in** $\mathrm{Struc}_{\mathcal{L}'}(M)$ if there is an injective G-map[1]

$$\Gamma : S'_x(M) \to \mathrm{Struc}_{\mathcal{L}'}(M).$$

We say that $S'_x(M)$ is **represented by** $S = \mathrm{Range}(\Gamma)$.

[1] i.e. a continuous map between the two compact topological spaces preserving the action of G.

The connection between IKMs and IREs

Let $\mathfrak{M}'_x(M)$ be the space of IKMs for $\mathcal{M}$ in the variable x whose support contains no realised type.

Corollary 10 (Braunfeld, Jahel and M. 2024)

Let $S'_x(M)$ be representable by S in $\mathrm{Struc}_{\mathcal{L}'}(M)$. Then, there is an isomorphism between $\mathfrak{M}'_x(M)$ and the space of IREs of $\mathcal{M}$ to $\mathcal{L}'$ concentrating on S.

Theorem 11 (Braunfeld, Jahel and M. 2024)

For every homogeneous $\mathcal{M}$, there is a language $\mathcal{L}^{pr}$ such that $S'_x(M)$ is representable in $\mathrm{Struc}_{\mathcal{L}^{pr}}(M)$.

To each relation in $\mathcal{L}$ we associate various relations of lower arity to code how x behaves with respect to $\mathcal{M}$.

Structure Independence

Definition 12 (Structure independence)

We say that an IRE μ of $\mathcal{M}$ to $\mathcal{L}'$ is **structure independent** if μ is actually S_∞-invariant.

Definition 13 (Age of a measure)

For an IRE μ of $\mathcal{M}$ to $\mathcal{L}'$ let

$$\mathrm{Age}(\mu) := \{\mathbf{H}^* \in \mathrm{Struc}_{\mathcal{L}'}(H) | H \in \mathrm{Age}(M), \mu(H^*) > 0\}.$$

Question 1

For which $\mathcal{M}$ and $\mathcal{F}'$ do we get that all IREs of $\mathcal{M}$ such that $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{F}'$ are structure independent?

Invariant Random Expansions with n-DAP I

Crane and Towsner 2018 study IREs of structures with **disjoint $\boldsymbol{n}$-amalgamation** for all n.

Definition 14 (Disjoint n-amalgamation)

Write $[n]$ for $\{1, \ldots, n\}$. For $I \subseteq [n]$, let $\mathcal{F}_I$ denote the set of structures in $\mathcal{F}$ with domain I.
A **disjoint $\boldsymbol{n}$-amalgamation problem:** for each $I \subseteq \{1, \ldots, n\}$ of size $n-1$, let $A_I \in \mathcal{F}_I$ be such that for $I \neq J$,

$$A_I \restriction (I \cap J) = A_J \restriction (I \cap J).$$

Solution: $A \in \mathcal{F}_{[n]}$ such that for each $I \subseteq \{1, \ldots, n\}$ of size $n-1$,

$$A \restriction I = A_I.$$

So $\mathcal{F}$ has **disjoint $\boldsymbol{n}$-amalgamation** (n-DAP) if all such problems have a solution.

Invariant Random Expansions with n-DAP II

- The random graph and universal homogeneous 3-hypergraph have n-DAP for all n;
- All other 3-hypergraphs we mentioned do not have 4-DAP;
- Crane and Towsner 2018 give a representation theorem for IREs of structures with n-DAP for all n similar to the Aldous-Hoover-Kallenberg one;
- In particular, if all relations of $\mathcal{L}'$ have **smaller arity than the smallest arity in** $\mathcal{L}$, then all IREs of $\mathcal{M}$ to $\mathcal{L}'$ are **structure independent**;
- By the IKM-IRE correspondence, the space of IKMs on the universal homogeneous k-hypergraph corresponds to the space of S_∞-invariant measures concentrating on the space of $(k-1)$-hypergraphs. This answers an open question of Albert 1994 and disproves a conjecture of Ensley 1996 (BJM 2024).

IREs to linear orders

What if instead we look at classes $\mathcal{F}'$ for which it is easy to prove that all IREs with $\mathrm{Age}(\mu)\restriction_{\mathcal{L}'} \subseteq \mathcal{F}'$ are structure independent?

Theorem 15 (Angel, Kechris, and Lyons 2014, rephrased)

Let $\mathcal{M}$ be a homogeneous hypergraph such that $\mathrm{Age}(M)$ has the free amalgamation property. There is a unique IRE μ of $\mathcal{M}$ to $\{<\}$ with $\mathrm{Age}(\mu)\restriction_{\{<\}} \subseteq \mathrm{LO}$, the space of finite linear orders. For $a_1, \ldots, a_n \in M$,

$$\mu(a_1 < \cdots < a_n) = \frac{1}{n!}.$$

- Note: they actually prove structure independence of any μ with $\mathrm{Age}(\mu)\restriction_{\{<\}} \subseteq \mathrm{LO}$!
- See Jahel and Tsankov 2022 for a strong generalisation.

From finite combinatorics to measures

The following Lemma was particularly inspiring to us.
For $\mathbf{H}, \mathbf{G}$ hypergraphs, write $N_{\mathrm{ind}}(\mathbf{H}, \mathbf{G})$ for the number of embeddings of $\mathbf{H}$ in $\mathbf{G}$. Given orderings $<_H$ and $<_G$ on $\mathbf{H}$ and $\mathbf{G}$ respectively, let $N_{\mathrm{ord}}(\mathbf{H}, \mathbf{G})$ denote the number of embeddings respecting the ordering.

Lemma 16 (cf. Lemma 2.1 in Angel, Kechris, and Lyons 2014)

Suppose that for all $\mathbf{H} \in \mathcal{F}_{[k]}$, and $\epsilon > 0$ there is $\mathbf{G} \in \mathcal{F}$ such that $N_{\mathrm{ind}}(\mathbf{H}, \mathbf{G}) > 0$ and for all orderings $<_G$ of $\mathbf{G}$ and $<_H$ of $\mathbf{H}$,

$$\left| \frac{N_{\mathrm{ord}}(\mathbf{H}, \mathbf{G})}{N_{\mathrm{ind}}(\mathbf{H}, \mathbf{G})} - \frac{1}{k!} \right| < \varepsilon.$$

Then, there is a unique IRE μ of $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$ such that $\mathrm{Age}(\mu) \restriction_{\{<\}} \subseteq \mathrm{LO}$.

Some notation for our results

Our setting: Two Fraïssé classes $\mathcal{F}$ and $\mathcal{F}'$ in languages $\mathcal{L}, \mathcal{L}'$. For $\mathbf{H} \in \mathcal{F}_{[n]}$ and $\mathbf{H}' \in \mathcal{F}'_{[n]}$, write $\mathbf{H} * \mathbf{H}'$ for the **free superposition** of the two structures. Write $\mathcal{F}^* = \mathcal{F} * \mathcal{F}'$ for the class of free superpositions of structures in $\mathcal{F}$ and $\mathcal{F}'$.
We have a homogeneous structure $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$. We study IREs μ of $\mathcal{M}$ to $\mathcal{L}'$ such that $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{F}'$.

Notation 17

Let $\mathbf{H}, \mathbf{G} \in \mathcal{F}$. Let Θ be a family of embeddings of $\mathbf{H}$ in $\mathbf{G}$. Let $\mathbf{H}^*, \mathbf{G}^* \in \mathcal{F}^*$ be such that $\mathbf{H}^*_{\restriction \mathcal{L}} = \mathbf{H}$ and $\mathbf{G}^*_{\restriction \mathcal{L}} = \mathbf{G}$. Then, we write

$$N_\Theta(\mathbf{H}^*, \mathbf{G}^*)$$

for the number of embeddings in Θ that are also embeddings of $\mathbf{H}^*$ in $\mathbf{G}^*$.

From finite combinatorics to structure independence

Lemma 18 (Braunfeld, Jahel and M. 2024)

Let $\mathbf{H}' \in \mathcal{F}'_{[k]}$ *and* $\mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[k]}$*. Suppose that for all* $\epsilon > 0$ *there is* $n > 0$ *and* $\mathbf{G} \in \mathcal{F}_{[n]}$ *with a family* Θ_i *of embeddings of* $\mathbf{H}_i$ *in* $\mathbf{G}$ *such that for all* $\mathbf{G}' \in \mathcal{F}'_{[n]}$*,*

$$\left| \frac{N_{\Theta_1}(\mathbf{H}_1^*, \mathbf{G}^*)}{|\Theta_1|} - \frac{N_{\Theta_2}(\mathbf{H}_2^*, \mathbf{G}^*)}{|\Theta_2|} \right| < \varepsilon, \qquad (\clubsuit)$$

where $\mathbf{G}^* := \mathbf{G} * \mathbf{G}', \mathbf{H}_i^* := \mathbf{H}_i * \mathbf{H}'$*.*
Then for any IRE μ *of* $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$ *such that* $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subset \mathcal{F}'$*,*

$$\mu(\mathbf{H}_1 * \mathbf{H}') = \mu(\mathbf{H}_2 * \mathbf{H}').$$

If we can do this for all $\mathbf{H}' \in \mathcal{F}'_{[k]}, \mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[k]}$, we get structure independence!

A strategy for structure independence

- **What we need:** Given $\mathbf{H}' \in \mathcal{F}'_{[k]}, \mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[k]}$, we want to find a large $\mathbf{G} \in \mathcal{F}$ and a family of embeddings Θ_i of $\mathbf{H}_i$ in $\mathbf{G}$ such that no matter how we add a structure from $\mathcal{F}'$ onto $\mathbf{G}$, there is still a similar proportion of copies of $\mathbf{H}_1 * \mathbf{H}'$ and $\mathbf{H}_2 * \mathbf{H}'$ (counting among the embeddings Θ_i);
- **Method:**
  - Find a large uniform k-hypergraph $\mathbb{K}$. We will need $\mathbb{K}$ to be dense enough. Moreover, it must be so that we can glue copies of $\mathbf{H}_1$ and $\mathbf{H}_2$ onto the hyperedges of $\mathbb{K}$ and still get a structure in $\mathcal{F}$;
  - Build $\mathbf{G}$ by gluing independently at random copies of $\mathbf{H}_1$ and $\mathbf{H}_2$ on the hyperedges of $\mathbb{K}$;
  - We will see that if the growth rate of structures in $\mathcal{F}'$ is slow enough, $\mathbf{G}$ will satisfy ($\clubsuit$).

k-overlap closedness

Definition 19 (k-overlap closed class)

Let $k <$ the minimal arity in $\mathcal{L}$. $\mathcal{F}$ is **k-overlap closed** if for all $r \geq k$ there is $N \geq r$ such that for all $n \geq N$ there is r-uniform hypergraph on n vertices satisfying the following conditions:

1. There are at least $C(r)n^{k+\epsilon(r)}$ many hyperedges;
2. No two hyperedges intersect in more than k points;
3. If structures from $\mathcal{F}_{[r]}$ are pasted into the hyperedges, the resulting structure is in $\mathcal{F}$.

- We will prove that having k-overlap closed age is sufficient to get structure independence for IREs to a k-ary language;
- To prove that $\mathcal{F}$ is k-overlap closed, we need to build large and dense enough r-uniform hypergraphs so that pasting the structures into hyperedges we are still avoiding all configurations forbidden in $\mathcal{F}$.

Some k-overlap closed classes

Using a probabilistic method argument we can show that the following free amalgamation classes in a $(k+1)$-ary language such that their Fraïssé limit has k-transitive automorphism group are k-overlap closed:

- The class of tetrahedron-free 3-hypergraphs is 2-overlap closed;
- Any free amalgamation class in a $(k+1)$-ary language $\mathcal{L}$ where all omitted substructures are $(k+1)$-irreducible is k-overlap closed. By Conant 2017, these are **simple**, and all ternary simple 2-transitive structures with free amalgamation are of this form;
- There are also various NSOP_4 free amalgamation classes for which the argument holds (e.g. the class of $\mathcal{K}_4^-$-free 3-hypergraphs);
- Any free amalgamation class of arity $(k+1)$ is 1-overlap closed. It is unclear whether they are also all k-overlap closed.

The main Theorem

Theorem 20 (Braunfeld, Jahel and M. 2024)

Let $\mathcal{F}$ be a k-overlap closed Fraïssé class and let $\mathcal{C}$ be a hereditary class of $\mathcal{L}'$-structures with labelled growth rate $O(e^{n^{k+\delta}})$ for every $\delta > 0$. Then any IRE μ of $\mathcal{M} = \mathrm{Flim}(\mathcal{F})$ such that $\mathrm{Age}(\mu) \restriction_{\mathcal{L}'} \subseteq \mathcal{C}$ is structure independent.

Proof idea.

Start with $\mathbf{H}_1, \mathbf{H}_2 \in \mathcal{F}_{[r]}$. Take $\mathbb{K}$ to be an r-uniform hypergraph on n vertices satisfying the conditions for k-overlap closedness. Build $\tilde{\mathbf{G}}$ by gluing with probability $1/2$ on the hyperedges of $\mathbb{K}$ copies of $\mathbf{H}_1$ and $\mathbf{H}_2$. Let Θ_i be the embeddings of $\mathbf{H}_i$ in $\tilde{\mathbf{G}}$ whose image is an r-hyperedge. Fix $\mathbf{H}' \in \mathcal{F}_{[r]}$ and $\mathbf{G}' \in \mathcal{F}_{[n]}$. We prove that

$$\mathbb{P}(\text{there is a } \mathbf{G}' \text{ such that } \clubsuit \text{ fails for } \tilde{\mathbf{G}}) \to 0 \text{ as } n \to \infty.$$

□

Consequences

Corollary 21 (Braunfeld, Jahel and M. 2024)

For the following 3-hypergraphs $\mathcal{M}$, any IRE μ of $\mathcal{M}$ to a binary language $\{E\}$ such that $\mathrm{Age}(\mu) \restriction_{\{E\}} \subseteq \mathrm{GRAPHS}$ is structure independent:

- *Any simple 3-hypergraph whose age has free amalgamation (this includes the universal homogeneous 3-hypergraph and the generic tetrahedron-free 3-hypergraph);*
- *The universal $\mathcal{K}_4^-$-free 3-hypergraph.*

More generally, if $\mathcal{M}$ has k-overlap closed age, any IRE by finitely many k-ary relations is structure independent.

An application to Keisler measures

Recently, people have been interested in comparing the following two sets of formulas (which capture a notion of 'smallness'):

- $F(\emptyset) :=$ formulas forking over $\emptyset$;
- $\mathcal{O}(\emptyset) :=$ formulas which are assigned measure zero by every IKM.

- $F(\emptyset) \subseteq \mathcal{O}(\emptyset)$ in any theory;
- $F(\emptyset) = \mathcal{O}(\emptyset)$ in stable theories;
- In Chernikov, Hrushovski, Kruckman, Krupinski, Moconja, Pillay, and Ramsey 2021, they give the first examples of simple theories where $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$;
- I found the first simple ω-categorical examples of $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ (Marimon 2023). However, these were not one-based;
- Some original motivation for studying structure independence is that if we could prove it for the generic tetrahedron-free 3-hypergraph, we would have a one-based simple example where $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$.

A dichotomy for Keisler measures I

Corollary 22 (Braunfeld, Jahel and M. 2024)

Let $\mathcal{M}$ be a simple homogeneous structure in a finite ternary language whose age has free amalgamation and such that $\mathrm{Aut}(M)$ is 2-transitive. Then, any invariant Keisler measure for $\mathcal{M}$ in x is structure independent. Moreover,

1. *EITHER: $\mathrm{Age}(M)$ has disjoint n-amalgamation for all n. In this case there is an IKM μ assigning positive measure to every non-forking formula;*
2. *OR: $\mathrm{Age}(M)$ fails disjoint n-amalgamation for some n. In this case,*

$$F(\emptyset) \subsetneq \mathcal{O}(\emptyset).$$

A dichotomy for Keisler measures II

Proof idea.

By Conant 2017, independence is trivial. Case 1 relies on such structure being essentially random. Case 2, relies on finding a formula $\phi(x, \overline{y})$ which, over some parameters A witnesses the failure of disjoint amalgamation (so $\phi(x, A)$ is inconsistent), but which is consistent (and so non-forking) over a different set of parameters. Then, by structure independence $\phi(x, A)$ must be assigned measure zero. □

When DO measures care about structure?

Our results also offer an heuristic for when we might expect there to be structure dependent IREs for expansions to lower arity languages (and so structure dependent IKMs).

In particular, if $\mathrm{Age}(M)$ has slow labelled growth compared to the arity of its language, we expect any similar strategy to fail.

For example, the class of two-graphs has slower growth than the class of graphs (and there is no way for it to be k-overlap closed either).

The two-graph

Theorem 23 (M. 2023, cf. Jahel 2021)

The universal homogeneous two-graph $\mathcal{G}$ has a unique invariant Keisler measure μ in the singleton variable. For $\phi(x, a_1, \ldots, a_n)$ isolating $\mathrm{tp}(d/a_1, \ldots, a_n)$, we have that

$$\mu(\phi(x, a_1, \ldots, a_n)) = \left(\frac{1}{2}\right)^{n-1},$$

and $\mu(x = x) = 1$.

- The proof exploits that, fixing a vertex, we can find a copy of the random graph canonically embedded in $\mathcal{G}$;
- A similar uniqueness result should hold for the higher arity versions of the two-graph (i.e. the kay-graphs).

Problems we are contemplating

- **What can we say about IREs of $\mathcal{M}$ to languages of larger arity?**
- Structure independence stops making sense, but there is a reasonable analogue in Crane and Towsner 2018 for which there is some hope;
- **Can we show that the IREs of any free amalgamation class to a language of lower arity are structure independent?**
- At the moment, it looks like one could have a structure with free-amalgamation and omitted substructures with sparse enough relations that our argument fails. But hopefully better combinatorial arguments can make even our proof-strategy work;
- **Can we say more about when and how structure independence fails?**
- Here I would be particularly interested in studying IKMs for ternary reducts of binary homogeneous structures.

Bibliography I

- Ackerman, Nathanael, Cameron Freer, Alex Kruckman, and Rehana Patel (2017). *Properly ergodic structures*. arXiv: `1710.09336 [math.LO]`.
- Ackerman, Nathanael, Cameron Freer, Aleksandra Kwiatkowska, and Rehana Patel (2017). "A classification of orbits admitting a unique invariant measure". In: *Annals of Pure and Applied Logic* 168.1, pp. 19–36.
- Ackerman, Nathanael, Cameron Freer, and Rehana Patel (2016). "Invariant measures concentrated on countable structures". In: *Forum of Mathematics, Sigma*. Vol. 4. Cambridge University Press, e17.
- Albert, Michael H (1994). "Measures on the random graph". In: *Journal of the London Mathematical Society* 50.3, pp. 417–429.

Bibliography II

- Aldous, David J (1981). "Representations for partially exchangeable arrays of random variables". In: *Journal of Multivariate Analysis* 11.4, pp. 581–598.
- Angel, Omer, Alexander S Kechris, and Russell Lyons (2014). "Random orderings and unique ergodicity of automorphism groups". In: *Journal of the European Mathematical Society* 16.10, pp. 2059–2095.
- Chernikov, Artem, Ehud Hrushovski, Alex Kruckman, Krzysztof Krupinski, Slavko Moconja, Anand Pillay, and Nicholas Ramsey (2021). *Invariant measures in simple and in small theories*. arXiv: `2105.07281 [math.LO]`.

Bibliography III

- Conant, Gabriel (2017). "An axiomatic approach to free amalgamation". In: *The Journal of Symbolic Logic* 82.2, pp. 648–671. ISSN: 00224812, 19435886. URL: http://www.jstor.org/stable/26358469 (visited on 02/08/2024).
- Crane, Harry and Henry Towsner (2018). "Relatively exchangeable structures". In: *The Journal of Symbolic Logic* 83.2, pp. 416–442.
- Ensley, Douglas E (1996). "Automorphism–invariant measures on $\aleph_0$-categorical structures without the independence property". In: *The Journal of Symbolic Logic* 61.2, pp. 640–652.
- Hoover, Douglas N (1979). "Relations on Probability Spaces and Arrays of". In: *t, Institute for Advanced Study*.
- Jahel, Colin (July 2021). "Some progress on the unique ergodicity problem". PhD thesis. Université Claude Bernard, Lyon 1. URL: http://math.univ-lyon1.fr/~jahel/doc/These.pdf.

Bibliography IV

- Jahel, Colin and Todor Tsankov (2022). "Invariant measures on products and on the space of linear orders". In: *Journal de l'École polytechnique — Mathématiques* 9, pp. 155–176.
- Kallenberg, Olav (1997). *Foundations of modern probability*. eng. Probability and its applications. New York, NY [u.a.]: Springer. ISBN: 0387949577.
- Marimon, Paolo (2023). *Invariant Keisler measures for omega-categorical structures*. arXiv: 2211.14628 [math.LO].