When are bounded arity polynomials enough?

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Joint work with Reinhard Pöschel

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- Certain relations called *higher dimensional equivalences*, which play a role in commutator theory, satisfy higher arity versions of Ξ. For a d-dimensional equivalence relation,

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$$\begin{array}{ccc} b & \text{——} & d \\ | & & | \\ a & \text{——} & c \end{array} \quad \text{and } (a, c), (c, d), (a, b), (b, d) \in \varrho \implies (a, d) \in \varrho$$

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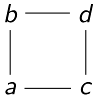

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Suppose $\rho \subseteq B^{2^1}$ is a quasiorder. We depict pairs in ρ as 'lines': $\begin{array}{c} a \\ | \\ b \end{array} \in \rho$

Let ρ_2 to be the set of all $\begin{array}{ccc} a & - & b \\ | & & | \\ c & - & d \end{array} \in B^{2^2}$ such that $\begin{array}{cccc} a & b & a & c \\ | & | & | & | \\ c & d & b & d \end{array} \in \rho$

In general, let $\rho_n \subseteq B^{2^n}$ be the set of all labeled hypercubes in which every edge determines a ρ-pair.

We therefore have a sequence of sets of functions
$\rho \subseteq B^{2^1}, \rho_2 \subseteq B^{2^2}, \ldots, \rho_n \subseteq B^{2^n}, \ldots$

The transitivity of ρ implies that this collection of functions is closed under variable identification.

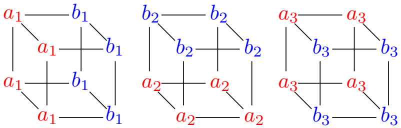
$$f(x,y) \in \rho_2 \qquad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & a \; - & b \\ & | \; \searrow & | \; \rho \\ 1 & c \; - & d \\ & \rho & \end{array} \qquad f(x,x) \in \rho \qquad \begin{array}{c|c} 0 & a \\ & | \\ 1 & d \end{array}$$

Aside: It's easy to see that this sequence of sets of functions is also closed under permutations of variables and the addition of dummy variables.

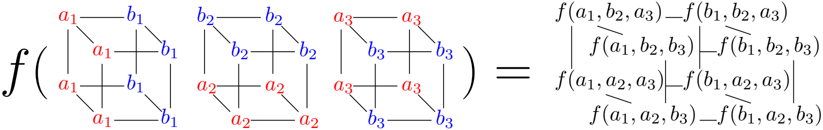
Suppose that $\mathrm{trl}_1(f) \triangleright \rho$. We want to see that $f \triangleright \rho$.

Take $f \in B^{B^3}$ (for example) and pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \rho$. We want to show that $(f(a_1, a_2, a_3), f(b_1, b_2, b_3)) \in \rho$.

Each of the following belongs to ρ_3:



Moreover, each line with the same position is labeled by an equality pair for two of the three cubes, hence the following belongs to ρ_3:



Call the above labeled cube $g(i_1, i_2, i_3)$. Identifying variables, we obtain $g(i_1, i_1, i_1) \in \rho$.

The following is a particular way of viewing this result:

Each part of the minion is defined by the property that any unary function obtained by evaluating some variables with constants belongs to ρ.

Special minion $\mathcal{M}$ with domain 2 and range X determined by quasiorder ρ:

$\rho \subseteq B^{2^1}$ $\quad \rho_2 \subseteq B^{2^2}$ $\quad \rho_3 \subseteq B^{2^3}$ $\quad \cdots$ $\rho_n \subseteq B^{2^n}$ $\quad \cdots$

$\mathrm{Pol}(\mathcal{M})$

$\mathrm{Pol}_1(\rho) \subseteq B^{B^1}$ $\quad \mathrm{Pol}_2(\rho) \subseteq B^{B^2}$ $\quad \cdots$ $\quad \mathrm{Pol}_n(\rho) \subseteq B^{B^n}$ $\quad \cdots$

This property is inherited by the polymoprhism clone, that is, $f \in \mathrm{Pol}(\mathcal{M})$ if and only if $\mathrm{trl}_1(f) \in \mathrm{Pol}_1(\mathcal{M})$.

This behavior is a special case of the more general situation.

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Let $f : B^{A^n}$ be a function. For $1 \leq d$, we say a function $g \in B^{A^d}$ is a *d-translation of f* if

$$g(x_1, \ldots, x_d) = f(\underbrace{c_1^1, \ldots, c_{k_1}^1}_{\text{constants}}, \mathbf{x_u}, \underbrace{c_1^2, \ldots, c_{k_2}^2}_{\text{constants}}, \mathbf{x_{u+1}}, \ldots, \underbrace{c_1^{l+1}, \ldots, c_{k_{l+1}}^{l+1}}_{\text{constants}}, \mathbf{x_{u+l}}, \underbrace{c_1^{l+2}, \ldots, c_{k_{l+2}}^{l+2}}_{\text{constants}})$$

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for some $1 \leq u \leq u + l \leq d$ and constants evaluated at other inputs. Set

$$\mathrm{trl}_d(f) = \{g : g \text{ is a } d\text{-translation of } f\}$$

and for $\varrho \subseteq B^{A^n}$

$$\mathrm{trl}_d(\varrho) = \bigcup \{\mathrm{trl}_d(f) : f \in \varrho\}.$$

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- This defines a sequence of sets of functions which we call ϱ^*:

$$\varrho_1 \subseteq B^{A^1}, \varrho_2 \subseteq B^{A^2}, \ldots, \underbrace{\varrho = \varrho_d \subseteq B^{A^d}}_{\text{original relation}}, \ldots, \varrho_n \subseteq B^{A^n}, \ldots$$

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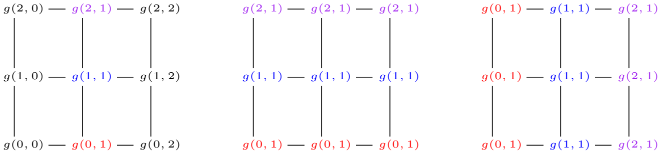
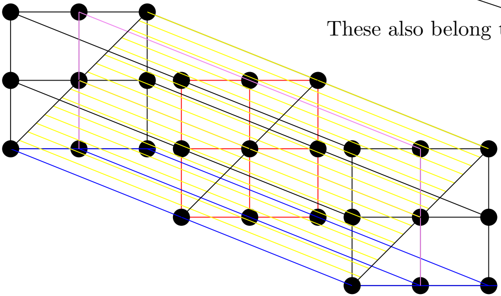
- If ϱ is both internally reflexive and transitive, then we call ϱ an *elementary type d-dimensional generalized quasiorder*.

Some pictures to provide intuition for the definitions:



$g(i_1, i_2) \in \rho \subseteq B^{A^2}$ $f(i_1, i_2) = g(i_1, 1)$ $h(i_1, i_2) = g(i_2, 1)$

These also belong to ρ if ρ is internally reflexive.



Transitivity means that if a ternary function has all 'slices' belonging to ρ, then the diagonal also belongs to ρ.

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Proposition

Let $\varrho \subseteq B^{A^d}$ be an elementary type d-dimensional GQuord*. Then ϱ satisfies the property Ξ_d: for every $1 \geq n$ and $f \in B^{B^n}$,*

$$f \triangleright \varrho \iff \mathrm{trl}_d(f) \triangleright \varrho.$$

Proof sketch ($d = 3, n = 4$).

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Notice that $f\left(a(i_1^1, i_2^1, i_3^1), b(i_1^2, i_2^2, i_3^2), c(i_1^3, i_2^3, i_3^3), d(i_1^4, i_2^4, i_3^4)\right) \in \varrho_9 \subseteq B^{A^9}$, because any way of substituting at least 9 constants above produces an element of $\mathrm{trl}_3(f)$.

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$f\left(a(i_1^1, i_2^1, i_3^1), b(i_1^2, i_2^2, i_3^2), c(i_1^3, i_2^3, i_3^3), d(i_1^4, i_2^4, i_3^4)\right) \in \varrho_9 \subseteq B^{A^9}$,
because any way of substituting at least 9 constants above produces an element of $\mathrm{trl}_3(f)$. For example, for constants $r_1, \ldots, r_9$:

$$f\left(a(\mathbf{i_1^1}, r_1, \mathbf{i_3^1}), b(\mathbf{i_1^2}, r_2, r_3), c(r_4, r_5, r_6), d(r_7, r_8, r_9)\right) \in \varrho \subseteq B^{A^3}$$

Proof sketch ($d = 3, n = 4$).

If $f \triangleright \varrho$, then the reflexivity of ϱ implies that $trl_3(f) \triangleright \varrho$. On the other hand, suppose $\mathrm{trl}_3(f) \triangleright \varrho$. Given $a(i_1, i_2, i_3), b(i_1, i_2, i_3), c(i_1, i_2, i_3), d(i_1, i_2, i_3) \in \varrho$, we want to show

$$f\left(a(i_1, i_2, i_3), b(i_1, i_2, i_3), c(i_1, i_2, i_3), d(i_1, i_2, i_3)\right) \in \varrho.$$

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Now we can apply the transitivity assumption and identify variables:

$$f\left(a(i_1^1, i_2^1, i_3^1), b(i_1^1, i_2^1, i_3^1), c(i_1^1, i_2^1, i_3^1), d(i_1^1, i_2^1, i_3^1)\right) \in \varrho \subseteq B^{A^3}$$

This generalizes what we showed for quasiorders:

ρ_n is the set of all $a \in B^{A^n}$ such that $\mathrm{trl}_d(a) \subseteq \rho$ for all $a \in \rho_n$.

Collection of functions ρ^* with domain A and range B determined by quasiorder ρ:

$\rho_1 \subseteq B^{A^1}$ $\cdots$ $\rho_d = \rho \subseteq B^{A^d}$ $\cdots$ $\rho_n \subseteq B^{A^n}$ $\cdots$

$\mathrm{Pol}(\rho) = \mathrm{Pol}_d(\rho)^*$ $\mathrm{Pol}_1(\rho) \subseteq B^{B^1}$ $\cdots$ $\mathrm{Pol}_d(\rho) \subseteq B^{B^d}$ $\cdots$ $\mathrm{Pol}_n(\rho) \subseteq B^{B^n}$ $\cdots$

This property is inherited by the polymoprhism clone, that is, $f \in \mathrm{Pol}(\rho)$ if and only if $\mathrm{trl}_d(f) \in \mathrm{Pol}_d(\rho)$.

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Note that for *any* $\varrho \subseteq B^{A^d}$ we can form the sequence

$$\varrho_1, \varrho_2, \dots, \varrho_{d-1}, \varrho, \varrho_{d+1}, \dots, \varrho_n, \dots$$

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- So, for a polynomial clone $\mathcal{C}$, we can define the *generalized quasiorder dimension* of $\mathcal{C}$ as the least d such that $\mathcal{C} = (\mathcal{C}_d)^*$ (set this dimension to ∞ if such a d does not exist).

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$$\varphi_i : B^{A^{d_1} \cdots \times A^{d_{i-1}} \times A^{d_i} \times A^{d_{i+1}} \cdots \times A^{d_s}} \to (B^{A^{d_1} \cdots \times A^{d_{i-1}} \times A^{d_{i+1}} \cdots \times A^{d_s}})^{A^{d_i}}$$

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Examples

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- On the two element domain $\{0, 1\}$ there are four one dimensional quasiorders of arity 2:

$$\begin{aligned} \Delta &= \{(0,0),(1,1)\} \\ \varrho_1 &= \Delta \cup \{(1,0)\} \\ \varrho_2 &= \Delta \cup \{(0,1)\} \\ \nabla &= \varrho_1 \cup \varrho_2 \end{aligned}$$

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- Only ϱ_2 and ∇ contain the projection operation. In the case of ϱ_2, we have:

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$$\wedge = \begin{array}{ccc} 0 & \text{---} & 1 \\ | & & | \\ 0 & \text{---} & 0 \end{array}, \vee = \begin{array}{ccc} 1 & \text{---} & 1 \\ | & & | \\ 0 & \text{---} & 1 \end{array} \in (\varrho_2)^*$$

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- Obviously, we have in the case of ∇:

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- Neither Δ or ϱ_1 contain the projection operation, hence neither $(\varrho_1)^*$ or $(\Delta)^*$ is a clone. For example,

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- On the other hand, we know that both clones $\mathrm{Pol}(\Delta)$ and $\mathrm{Pol}(\varrho_1)$ have dimension 1, hence each is equal to one of the two clones from earlier. In this case:

$$\mathrm{Pol}(\varrho_1) = \mathrm{Clo}(\langle\{0,1\}; \wedge, \vee, 0, 1\rangle)$$
$$\mathrm{Pol}(\Delta) = \mathrm{Clo}(\langle\{0,1\}; \wedge, \vee, \neg, 0, 1\rangle).$$

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$$\varrho_L = \left\{ \begin{array}{ccc} y & \text{---} & w \\ | & & | \\ x & \text{---} & z \end{array} \in 2^{2^2} : x + y + z + w \equiv 0 \mod 2 \right\}.$$

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- The other 3 polynomial clones on $\{0, 1\}$ are also (2)-dimensional.

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$$f(x_1,\ldots,x_{d+1}) = \begin{cases} 0 & \text{if } x_1 + \cdots + x_{d+1} \leq d+1 \\ 2 & \text{if } x_1 = \cdots = x_{d+1} = 2 \\ 1 & \text{otherwise.} \end{cases}$$

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- Can show that $f \in (\mathcal{C}_d)^*$, but $f(x, x, \ldots, x)$ is not.

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- On the other hand, $\mathcal{C} = (\mathcal{C}_3)^*$, so $\mathcal{C}$ has dimension (3).
- In general, the algebra $\mathbb{A}_d = \langle \{0, 1, 2, 3\}; 2x_1 \ldots x_d, 0, 1, 2, 3\rangle$ determines a clone with dimension $d + 1$.

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- For example, let $\mathbb{A}$ be a Maltsev algebra and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$. Set $M(\alpha, \beta)$ to be

$$\mathrm{Sg}_{\mathbb{A}^{2^2}}\left(\left\{\begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array} : \langle x, y\rangle \in \alpha\right\} \cup \left\{\begin{array}{ccc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} : \langle x, y\rangle \in \beta\right\}\right)$$

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The commutator can then be defined as

$$[\alpha, \beta] = \left\{\langle x, y\rangle : \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} \in M(\alpha, \beta)\right\}$$

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It's easy to show that $M(\alpha, \beta)$ is a compound type (2)-dimensional generalized quasiorder. Furthermore, if $\alpha \neq \beta$, then $M(\alpha, \beta)$ is not elementary type.

- There also exist elementary type generalized quasiorders that are not compound type. Let $B = \{a, b, c\}$.

$$\varrho = \left\{ \begin{array}{ccc} y & \text{——} & u \\ | & & | \\ x & \text{——} & z \end{array} \in B^{2^2} : x = u \text{ or } y = z \text{ implies } x = y = u = z \right\}$$

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- ϱ is not a compound type GQuord, as witnessed by the following two elements:

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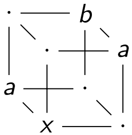
- ϱ is elementary type, because any way of filling in the following cube so that all faces belong to ϱ forces $a = b$.



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- The condition that a variety $\mathcal{V}$ is congruence meet-semidistributive is equivalent to the condition that the commutator is *neutral* for all congruences across the variety. This is equivalent to the collapse of certain intervals in higher dimensional congruence lattices. Which analogous intervals collapse in generalized quasiorder lattices for congruence meet-semidistributive varieties?

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- We hope to eventually apply this theory to say something about the lattice of clones on a finite set. For any clone $\mathcal{C}$, there is an infinite descending chain of clones

$$\mathrm{Pol}(\underbrace{\mathrm{GQuord}^{(1)}}_{(1)\text{-dimensional}}(\mathcal{C})) \geq \cdots \geq \mathrm{Pol}(\underbrace{\mathrm{GQuord}^{(n)}}_{(n)\text{-dimensional}}(\mathcal{C})) \geq \cdots \geq \mathcal{C}$$

Thank you for your attention!