CONSTRAINT SATISFACTION PROBLEMS

DISTINGUISHING THE EASY FROM THE HARD

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EUROPEAN RESEARCH COUNCIL

ERC SYNERGY GRANT POCOCOP (GA 101071674)
FWF I5948

HO!

OUTLINE

1. CONSTRAINT SATISFACTION PROBLEMS AND RELATIONAL STRUCTURES — 10 MINUTES
2. ALGEBRAIC INVARIANTS (FOR FINITE STRUCTURES) — 10 MINUTES
3. GOING INFINITE — 10 MINUTES
4. GOING FINITE — 10 MINUTES

50 MINUTES

OUTLINE

1. CONSTRAINT SATISFACTION PROBLEMS AND RELATIONAL STRUCTURES — 10 MINUTES
2. ALGEBRAIC INVARIANTS (FOR FINITE STRUCTURES) — 10 MINUTES
3. GOING INFINITE — 10 MINUTES
4. GOING FINITE — 10 MINUTES

~~50~~ 80 MINUTES

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$ SATISFYING ALL CONSTRAINTS ?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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MODEL:

- SET V OF POSSIBLE VALUES (FIXED)

  E.G. $\{0,1\}, \{0,1,2\}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \dots$

- ALLOWED CONSTRAINTS (FIXED)

  $C_1, \dots, C_m$ RELATIONS ON V

  $C_i \subseteq V^{a_i}$

SO $\mathbb{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE

= "TEMPLATE"

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- PP-SENTENCE

$$\varphi = \exists x_1 \dots \exists x_n \; C_{i_1}(\text{VARIABLES}) \wedge C_{i_2}(\text{VARIABLES}) \wedge \dots \; C_{i_\ell}(\text{VARIABLES})$$

QUESTION $\mathbb{A} \models \varphi$?

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QUESTION $\mathbb{A} \models \varphi$?

META-QUESTION

FOR WHAT $\mathbb{A}$ IS CSP($\mathbb{A}$) EASY / HARD ?

P: $\exists$ ALGORITHM PROVIDING ANSWER IN $p(\#\text{VARIABLES})$ STEPS, p POLYNOMIAL

NP: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLES

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1^2 + x_2^3 = x_3^4$

  $\vdots$

QUESTION
- SOLUTION IN $\mathbb{Z}$?

<u>UNDECIDABLE</u> (MATIYASEVIC '77)
HILBERT 10

EXAMPLES

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- SAME OVER $\mathbb{Z}_p$:

NP-COMPLETE

$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

EXAMPLES

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IN P (GAUSS)

EXAMPLES

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IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$)

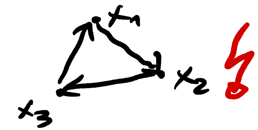
GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$

  $x_2 < x_3$

  $x_3 < x_1$

  $\vdots$

QUESTION
- SOLUTION IN $\mathbb{Q}$?



IN P

EXAMPLES

TERNARY RELATIONS

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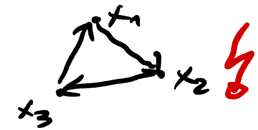
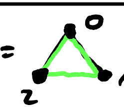


IN P

• $\mathbb{A} = K_3 =$ 

$= (\{0,1,2\}, E)$

CSP($\mathbb{A}$) **GIVEN** • VARIABLES $x_1 \dots x_n$

• CONSTRAINTS $E(x_1, x_2)$
$E(x_2, x_3)$
$\vdots$

QUESTION • SOLUTION IN $\{0,1,2\}$?

3-COLORING PROBLEM = NP-COMPLETE

THEOREM (BULATOV, ZHUK '17)
(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P OR

NP-COMPLETE

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NP-COMPLETE

(P $\neq$ NP)

CSP($\mathbb{A}$) $\in$ P $\Leftrightarrow$ $\mathbb{A}$ HAS ALGEBRAIC INVARIANT

$s(x,y,x,z,y,z) =$
$s(y,x,z,x,z,y)$

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ALGEBRAIC INVARIANTS = POLYMORPHISMS

$\mathbb{A} = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^\ell \to A$ POLYMORPHISM $:\Leftrightarrow$

f HOMOMORPHISM $\mathbb{A}^\ell \to \mathbb{A}$ $\Leftrightarrow$

$\forall i: \forall r_1 \dots r_\ell \in R_i$

$$f\left(\begin{pmatrix} \\ \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ \\ \\ \end{pmatrix}\right) \in R_i$$

$\boxed{\mathrm{Pol}(\mathbb{A})} := \{ f \mid f \text{ POLYMORPHISM OF } \mathbb{A} \}$

- CONTAINS PROJECTIONS $(x_1 \dots x_\ell) \mapsto x_j$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS OF AN ALGEBRA ON A !

EXAMPLES

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \in \mathrm{Pol}(\mathbb{Q}, <)$
- $(x,y,z) \mapsto x - y + z \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
- $? \in \mathrm{Pol}(K_3)$

EXAMPLES

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(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69)
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \Rightarrow \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\Rightarrow$ CSP($\mathbb{A}$) HARDER THAN CSP($\mathbb{B}$)

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- GOOD FOR CLASSIFYING CSPs ON $\{0,1\}$
(SCHAEFER '78)

- BETTER : EQUATIONAL CONDITIONS:
  - $\exists f \in \mathrm{Pol}(\mathbb{Q}, <) \ \forall x,y \quad f(x,y) = f(y,x)$
  - $\exists f \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
    $\forall x,y,z \quad f(x,x,y) = f(y,x,x) = y$
  - $\nexists f \in \mathrm{Pol}(K_3) : \forall x \ f(f(f(x))) = x$ ☹

EXAMPLES

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$
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DEFINITION

- EQUATIONAL (STRONG MALCEV) CONDITION: SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$
$$s_1(\text{variables}) = t_1(\text{variables})$$
$$\wedge$$
$$\vdots$$
$$\wedge \ s_k(\text{variables}) = t_k(\text{variables})$$

WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL $:\Leftrightarrow \mathrm{Pol}(\text{ANY STRUCTURE}) \models \Psi$
  $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) := \{\Psi \mid \mathrm{Pol}(\mathbb{A}) \models \Psi\}$

$\boxed{\text{(ADOLESCENT) THEOREM}}$ (BULATOV + JEAVONS + KROKHIN '00)

$$EQ(Pol(\mathbb{A})) \subseteq EQ(Pol(\mathbb{B}))$$

$\Rightarrow$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

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(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

$\cancel{s_i}$ $\cancel{t_j}$

$\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}^1(\mathrm{Pol}(\mathbb{B}))$

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EQ(Pol(𝔸)) ⊆ EQ(Pol(𝔹))

⟹ 𝔸 PP-INTERPRETS 𝔹

⟹ CSP(𝔸) HARDER THAN CSP(𝔹)

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$\cancel{s}$ $\cancel{t}$

EQ^(Pol(𝔸)) ⊆ EQ^(Pol(𝔹))

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⟹ CSP(𝔸) HARDER THAN CSP(𝔹)

COROLLARY

EQ^(Pol(𝔸)) TRIVIAL

⟹ 𝔸 PP-CONSTRUCTS EVERYTHING

⟹ CSP(𝔸) NP-COMPLETE

(ADOLESCENT) THEOREM (BULATOV + JEAVONS + KROKHIN '00)

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COROLLARY

$\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP- CONSTRUCTS EVERYTHING

$\Rightarrow$ CSP($\mathbb{A}$) NP-COMPLETE

THEOREM $\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow$

$\mathrm{Pol}(\mathbb{A}) \models \exists f\ \forall x,y,z \quad f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

(SIGGERS '10)

$\models \exists f\ \forall a,r,e \quad f(a,r,e,a) = f(r,a,r,e)$

(KEARNES + MARKOVIĆ + MCKENZIE '14)

$\models \exists f\ \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall\, n \geq |A|$ PRIME

(BARTO + KOZIK '11)

$\models \exists f\ \forall x,y \quad f(x \dots x y) = \dots = f(y x \dots x)$

$\forall\, n \geq |A|$ PRIME

(MARÓTI + MCKENZIE '08)

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P (BULATOV, ZHUK '17)

GOING INFINITE

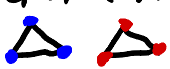
FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

E.G. 3-COLORING

INFINITE-DOMAIN CSPs:

EXAMPLES

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, \mathrm{Betw}(x,y,z))$
- GENERAL: UNBOUNDED SCHEDULING
- NO MONOCHROMATIC $\triangle$:
  - COLORING VERTICES OF GRAPH AVOIDING



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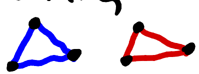

- GRAPH ORIENTATION AVOIDING

- $(\mathbb{Z}, +, 1, <)$
- $(\mathbb{Z}, +, 1, \cdot)$
- $(\mathbb{Q}, +, 1, \cdot)$

NUMERIC PROBLEMS

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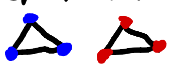
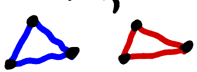
  - COLORING VERTICES OF GRAPH AVOIDING 
  - COLORING EDGES AVOIDING 
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- $(\mathbb{Z},+,1,<)$
- $(\mathbb{Z},+,1,\cdot)$
- $(\mathbb{Q},+,1,\cdot)$

NUMERIC PROBLEMS

"FORBIDDEN PATTERN PROBLEMS"

τ ... SIGNATURE E.G. $\{<\}$

$\mathfrak{F}$ ... FINITE SET OF "FORBIDDEN" τ-STRUCTURES

E.G. $\mathfrak{F} = \{$  $\}$

- GIVEN VARIABLES, CONSTRAINTS DEFINABLE OVER τ

  E.G. $Betw(x,y,z) \equiv x<y<z \vee z<y<x$
- $\exists$ SOLUTION IN SOME τ-STRUCTURE AVOIDING $\mathfrak{F}$?

GOING INFINITE

FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

E.G. 3-COLORING

INFINITE-DOMAIN CSPs:

EXAMPLES

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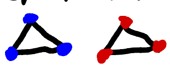

  - COLORING EDGES AVOIDING

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- $(\mathbb{Z}, +, 1, <)$
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OFTEN THERE IS "NICE" $\mathbb{A}$ WHOSE CSP IS THIS PROBLEM:

GOING INFINITE

FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

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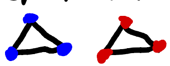

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NUMERIC PROBLEMS

"FORBIDDEN PATTERN PROBLEMS"

τ ... SIGNATURE E.G. $\{<\}$

$\mathcal{F}$ ... FINITE SET OF "FORBIDDEN" τ-STRUCTURES

E.G. $\mathcal{F} = \{$  $\}$

- GIVEN VARIABLES, CONSTRAINTS DEFINABLE OVER τ

  E.G. $\mathrm{Betw}(x,y,z) \equiv x<y<z \vee z<y<x$
- $\exists$ SOLUTION IN SOME τ-STRUCTURE AVOIDING $\mathcal{F}$?

OFTEN THERE IS "NICE" $\mathbb{A}$ WHOSE CSP IS THIS PROBLEM:

RELATIONS OF $\mathbb{A}$ DEFINABLE IN $\mathbb{B}$ S.T.

- $\mathrm{Aut}\,\mathbb{B} \curvearrowright B^n$ HAS FINITELY MANY ORBITS:

  $\mathbb{B}$ ω-CATEGORICAL
- $\exists d\ \forall k \geq d$ k-ORBITS UNIQUELY DETERMINED BY d-SUBORBITS

  $\mathbb{B}$ d-HOMOGENEOUS

  d-BOUNDED

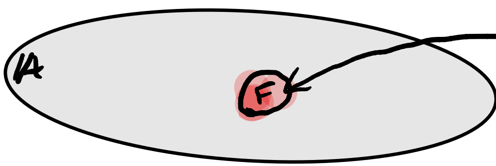
(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

$\mathbb{A}$ ω-CATEGORICAL

$EQ^{\wedge}_{\text{LOCAL}}(Pol(\mathbb{A}))$ TRIVIAL $\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ CSP($\mathbb{A}$) NP-HARD

$\exists F \subseteq A$ FINITE: $EQ^{\wedge}(Pol(\mathbb{A})|_F)$ TRIVIAL



Pol($\mathbb{A}$) NAKED

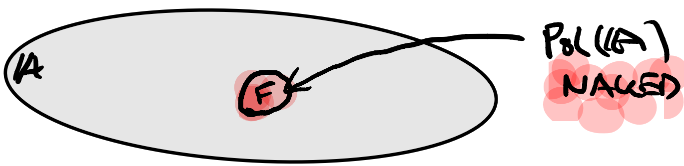
(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

$\mathbb{A}$ ω-categorical

$EQ^{\wedge}_{\text{local}}(Pol(\mathbb{A}))$ trivial $\Rightarrow$ $\mathbb{A}$ pp-constructs everything

$\Rightarrow$ $CSP(\mathbb{A})$ NP-hard

$\exists F \subseteq A$ finite: $EQ^{\wedge}(Pol(\mathbb{A})|_F)$ trivial



THEOREM

$\mathbb{A}$ ω-categorical

$EQ^{\wedge}_{\text{local}}(Pol(\mathbb{A}))$ non-trivial

$\Rightarrow$

$$Pol(\mathbb{A}) \models \exists u,v,f \ \forall x,y,z$$

$$u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$$

(BARTO + P. '17)

$\models$ ... many more

(BARTO + BODOR + KOZIK + MOTTET + P. '23)

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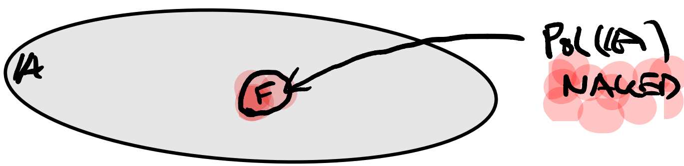

THEOREM

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GOING ~~IN~~FINITE

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... SO WHAT ?

PROBLEM: f MIGHT BE UGLY !

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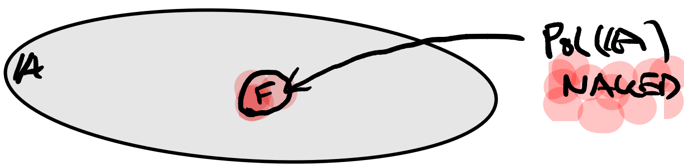

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... SO WHAT?

PROBLEM: f MIGHT BE UGLY!

UGLINESS: IF $\mathbb{A}$ IS DEFINABLE IN d-HOMOGENEOUS $\mathbb{B}$

$CSP(\mathbb{A})$: GIVEN $x_1 \dots x_n$
WANT TO ASSIGN $Aut(\mathbb{B})$-ORBITS
TO d-TUPLES OF VARIABLES

... BUT f MIGHT NOT ACT ON ORBITS

EXAMPLE $CSP(\mathbb{Q}, Betw(x,y,z))$

GIVEN VARIABLES $x_1, \dots, x_n$
WANT TO ASSIGN "<", ">", "="
TO EVERY PAIR (x_i, x_j)

DEFINITION $\mathbb{B}$ d-HOMOGENEOUS

$f : B^d \to B$ CANONICAL WRT $\mathbb{B}$ $:\Longleftrightarrow$

f PRESERVES EQUIVALENCE RELATION INDUCED BY $\mathrm{Aut}(\mathbb{B}) \curvearrowright B^d$

EXAMPLE

- $f : \mathbb{Q} \to \mathbb{Q}$, $x \mapsto -x$ CANONICAL WRT $(\mathbb{Q}, <)$
- $\min : \mathbb{Q}^2 \to \mathbb{Q}$ <u>NOT</u> CANONICAL

$$\begin{array}{ccccc} \min(a, & b) & = & u \\ \vee & \wedge & & ? \\ \min(c, & d) & = & v \end{array}$$

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THEOREM (MOTTET + BODIRSKY '18)

$\mathbb{A}$ DEFINABLE IN $\mathbb{B}$, $\mathbb{B}$ d-HOMOGENEOUS, d-BOUNDED

$f \in \mathrm{Pol}(\mathbb{A})$ CANONICAL WRT $\mathbb{B}$,

$u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in P$

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- BODIRSKY + TSANKOV + P. '13
  USE RAMSEY THEORY TO FIND CANONICAL POLYMORPHISMS (THAT MIGHT NOT SATISFY IDENTITIES)

- MOTTET + P. '22
  USE ALGEBRA TO FIND CANONICAL POLYMORPHISMS SATISFYING IDENTITIES

THEOREM

LET $\mathbb{A}$ BE DEFINABLE IN:

- $\mathbb{Q}$ (BODIRSKY + KÁRA '07)
- THE RANDOM GRAPH (BODIRSKY + P. '11)
- ANY HOMOGENEOUS GRAPH (BODIRSKY + MARTIN + PONGRÁCZ + P. '16)
- THE UNIVERSAL HOMOGENEOUS TOURNAMENT (MOTTET + P. '20)
- PARTIAL ORDER (KOMPATSCHER + VAN PHAM '17)
- 2-BRANCHING C-RELATION (BODIRSKY + JONSSON + VAN PHAM '16)
- ANY FINITELY BOUNDED HOMOGENEOUS HYPERGRAPH WITH RAMSEY EXPANSION BY GENERIC TOTAL ORDER (MOTTET + NAGY + P. '23)

- IF $\mathrm{Pol}(\mathbb{A}) \models \exists u,v,f\ \forall x,y,z \quad u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$ THEN $\mathrm{CSP}(\mathbb{A}) \in P$
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CONJECTURE

(BODIRSKY + P. '11, BARTO + P. '16)

TRUE $\forall \mathbb{A}$ DEFINABLE IN d-BOUNDED HOMOGENEOUS $\mathbb{B}$

THANK YOU !

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