# RECONSTRUCTING THE TOPOLOGY OF ALGEBRAIC STRUCTURES: HOW & WHY


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TU WIEN


European Research Council
Established by the European Commission


SyG POCOCOP
GA 101071674

FWF Austrian Science Fund
I 5948


SUMTOPO 2024
COIMBRA

## PART I : MOTIVATION & RESULTS

- ALGEBRAIC - TOPOLOGICAL STRUCTURES
- AUTOMORPHISM GROUPS, ENDOMORPHISM MONOIDS, POLYMORPHISM CLONES
- RECONSTRUCTION NOTIONS
- THE SCOREBOARD

## PART II : METHODS

- THE ZARISKI TOPOLOGY
- HOMOMORPHISM GLUING & HOMOGENEITY

MANY MATHEMATICAL OBJECTS CARRY:

- ALGEBRAIC STRUCTURE
- TOPOLOGICAL STRUCTURE

COMPATIBLE: ALGEBRAIC OPERATIONS CONTINUOUS.

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- ALGEBRA $(A; (f_i)_{i\in I})$: COMPOSITION + POINTWISE CONVERGENCE.

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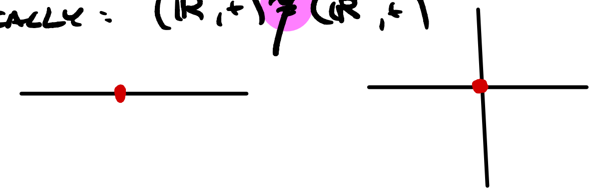

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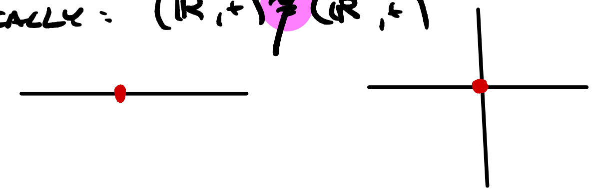

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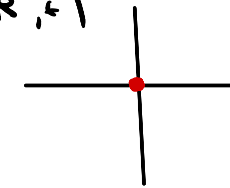

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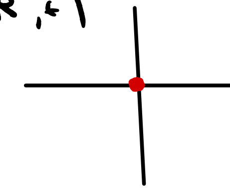

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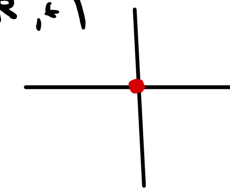

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**QUESTION** IS THERE A UNIQUE . . . TOPOLOGY?

. . . E.G.
- $T_2$ / HAUSDORFF
- SEPARABLE
- POLISH = SEPARABLE + COMPLETELY METRIZABLE

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**ANSWER** NO:

- EVERY $T_1$ TOPOLOGY $\supseteq$ PW (GAUGHAN '70s)
- $\mathcal{J} \subseteq \mathcal{J}'$, BOTH POLISH $\Rightarrow$ $\mathcal{J} = \mathcal{J}'$

$\Rightarrow$ PW UNIQUE POLISH

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**UNIQUE POLISH TOPOLOGY**:

- ISOMETRY GROUP OF THE URYSOHN SPACE / SPHERE (SABOK '13)
- HOMEOMORPHISM GROUPS OF $[0,1]^{\mathbb{N}}$, $2^{\mathbb{N}}$ (KALLMAN '80s)
- $\mathrm{Aut}(\mathbb{Q},<)$ (HODGES, HODKINSON, LASCAR, SHELAH '93 / SOLECKI + ROSENDAL '07)
- $\mathrm{Aut}(G)$ $G$ ... RANDOM GRAPH (HRUSHOVSKI '92 / KECHRIS + ROSENDAL '07)

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$M_n := \{ f \in \mathbb{N}^{\mathbb{N}} \mid |\mathbb{N} \setminus f(\mathbb{N})| = n \}$ $(n \leq \omega)$

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**INBETWEEN:** RESTRICT $n$

(NO CLASSIFICATION)

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SIAM J. COMPUT.
Vol. 49, No. 2, pp. 365–393



# TOPOLOGY IS IRRELEVANT (IN A DICHOTOMY CONJECTURE FOR INFINITE DOMAIN CONSTRAINT SATISFACTION PROBLEMS)*

LIBOR BARTO† AND MICHAEL PINSKER‡

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**EXAMPLES** LET $\mathbb{A}, \mathbb{B}$ BE COUNTABLE <u>$\omega$-CATEGORICAL</u>

- $\mathbb{A}, \mathbb{B}$ FIRST-ORDER BI-INTERPRETABLE $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ (AHLBRANDT ZIEGLER '80s)
  IF $\mathbb{A}$ HAS UNIQUE POLISH TOP.: $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ COQUAND '81
- $\mathbb{A}, \mathbb{B}$ EXISTENTIAL-POSITIVE BI-INT. $\Leftrightarrow$ $\mathrm{End}(\mathbb{A}) \cong \mathrm{End}(\mathbb{B})$ (BODIRSKY + JUNKER '08)
- $\mathbb{A}, \mathbb{B}$ PRIMITIVE-POSITIVE BI-INT. $\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \cong \mathrm{Pol}(\mathbb{B})$

(BODIRSKY + P. '11)

**DEFINITION**: $\mathbb{A}$ COUNTABLE **$\omega$-CATEGORICAL** $:\Leftrightarrow$

$\forall n \geq 1$ $\mathrm{Aut}(\mathbb{A}) \curvearrowright \mathbb{A}^n$ HAS FINITELY MANY ORBITS

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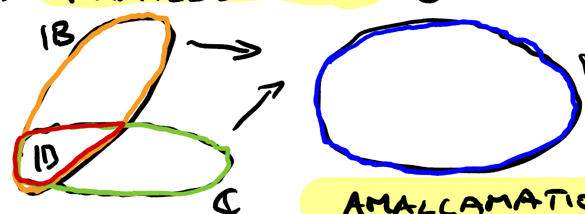


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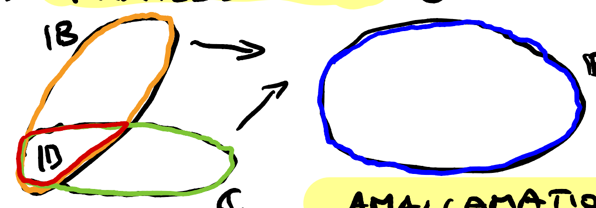


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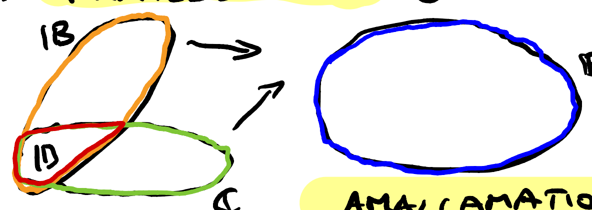


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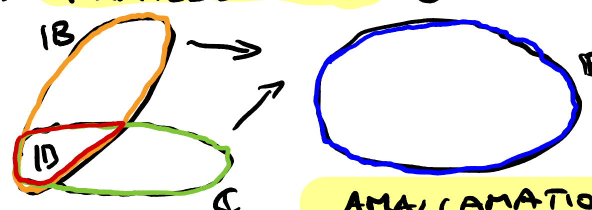


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- EVERYTHING DEFINABLE IN THOSE E.G. $(\mathbb{Q}, \mathrm{Betw}(x,y,z))$

VARIANT

AUTOMATIC CONTINUITY AC

$\mathrm{Aut}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\forall \varphi : \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N}) :$ $\varphi$ CONTINUOUS

$\mathrm{End}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

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MEANING FOR $\omega$-CATEGORICAL $\mathbb{A}$:

$\mathbb{B}$ HAS FO-INTERPRETATION IN $\mathbb{A}$ $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$ (MORE OR LESS)

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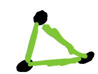
**EXAMPLE** CAN $\mathbb{A}$ PP-INTERPRET THE GRAPH $K_3$?



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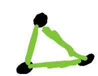
**EXAMPLE** CAN $\mathbb{A}$ PP-INTERPRET THE GRAPH $K_3$? 

**THEOREM** $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(\triangle)$ $\Leftrightarrow$ $\exists\, e, f, s \in \mathrm{Pol}(\mathbb{A})$:

$$\forall x,y,z \quad e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$$

(BARTO + P. '16, NOT TRUE AS STATED)

**RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$**
**$\mathbb{A}$... COUNTABLE $\omega$-CATEGORICAL**

THREE NOTIONS:

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$
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THREE NOTIONS:

UP: PW IS UNIQUE POLISH TOPOLOGY ON $\underbrace{\mathrm{Aut}(\mathbb{A})}_{\text{HARD TO FAIL}}$ / $\underbrace{\mathrm{End}(\mathbb{A}) / \mathrm{Pol}(\mathbb{A})}_{\text{FAILS EASILY}}$

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$
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---

AC: EVERY $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ } HARD TO FAIL
IS CONTINUOUS

FAILS EASILY {
$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$
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(AUTOMATIC CONTINUITY)

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## RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$
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(AUTOMATIC CONTINUITY)

---

**AH**: EVERY ISOMORPHISM $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$ IS A HOMEOMORPHISM

HARD TO FAIL {
$\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$
$\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

(AUTOMATIC HOMEOMORPHICITY)

EVANS + HEWITT '90
+ BODIRSKY + KOMPATSCHER + P. '15

## RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$
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HARD TO FAIL

EVANS + HEWITT '90
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---

**AR**: EVERY ISOMORPHISM $\varphi$ IS INDUCED BY BIJECTION $f: \mathbb{A} \to \mathbb{B}$

(ACTION RECONSTRUCTION)

---

$$\begin{tikzcd} & & AC \arrow[dl, dashed] \\ UP \arrow[r] & AH & \\ & & AR \arrow[ul, dotted] \end{tikzcd}$$

FOR GROUPS (OPEN FOR MONOIDS)

AR USUALLY RESTRICTED TO $\mathbb{B}$ SATISFYING CONDITIONS : NO ALGEBRAI

# THE SCOREBOARD

(INCOMPLETE + PROBABLY INCORRECT)

| | UNIQUE POLISH (UP) | AUTOMATIC CONT. (AC) | ACTION REC. (AR) | AUTOMATIC HOMEO (AH) |
|---|---|---|---|---|
| Aut | $(\mathbb{N},=)$ Gaughan '70s<br>$(\mathbb{Q},<)$ Rosendal + Solecki '07<br>$\mathcal{G}$ Kechris + Rosendal '07 | $(\mathbb{N},=)$ Dixon, Neumann, Thomas '86; Semmes '81<br>$(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89<br>$\mathcal{G}$, $\omega$-stable Hodges + Hodkinson + Lascar + Shelah '80s<br>$\mathbb{H}_n$ Herwig '80s | $(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94<br>Hypergraphs, Henson digraphs Barbina + Macpherson '07 | $(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 |
| End | $\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{Q},\leq)$ Schmid + P. '23<br>$(\mathbb{N},=)$ EJMMMP '20 | $\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{N},=)$ EJMMMP '20 | $(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19 | $(\mathbb{N},\neq)$, $\mathcal{G}$, $\mathbb{T}$ Bodirsky + P. + Pongrácz '13<br>$(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17 |
| Pol | $\mathcal{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{N},=)$ Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '20 | $\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{N},=)$ EJMMMP '20 | $(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19 | $\supseteq \mathbb{N}^{\mathbb{N}}$, $\mathcal{G}$ Bodirsky + P. + Pongrácz '13<br>$(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Pech + Pech '18<br>$(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17<br>$(\mathbb{Q},<)$ Behrisch + Vargas-García '19 |

$\mathcal{G}$ ... RANDOM GRAPH

$\mathbb{P}$ ... RANDOM POSET

$\mathbb{T}$ ... RANDOM TOURNAMENT

$\mathbb{D}$ ... RANDOM DIGRAPH

$\mathbb{E}$ ... RANDOM EQUIVALENCE RELATION

$\mathbb{H}_n$ ... RANDOM $K_n$-FREE GRAPH

$\mathbb{B}$ ... COUNTABLE ATOMLESS BOOLEAN ALG

$\omega$-CAT

NO ALGEBRAICITY

## PART I : MOTIVATION & RESULTS

- ALGEBRAIC-TOPOLOGICAL STRUCTURES
- AUTOMORPHISM GROUPS, ENDOMORPHISM MONOIDS, POLYMORPHISM CLONES
- RECONSTRUCTION NOTIONS
- THE SCOREBOARD

## PART II : METHODS

- THE ZARISKI TOPOLOGY
- HOMOMORPHISM GLUING & HOMOGENEITY

# ZARISKI TOPOLOGY

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- CLOSED SETS: ALGEBRAICALLY GIVEN BY $\{s \mid a_1 \circ s \circ a_2 \circ \dots \circ a_n \circ s = b_1 \circ s \circ b_2 \circ \dots \circ b_m \circ s\}$
  (AS OPPOSED TO PW)
- ANY $T_1$ SEMIGROUP TOPOLOGY $\supseteq$ ZARISKI
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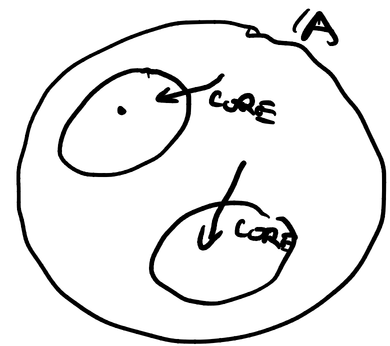
**THEOREM** (P. + SCHINDLER '23)

$\mathbb{A}$ $\omega$-CATEGORICAL, NO ALGEBRAICITY

- EVERY ELEMENT CONTAINED IN CORE
- CORE FINITE OR NO ALGEBRAICITY

$\Rightarrow$ PW = ZARISKI ON $\mathrm{End}(\mathbb{A})$




IN PARTICULAR ALL ON SCOREBOARD

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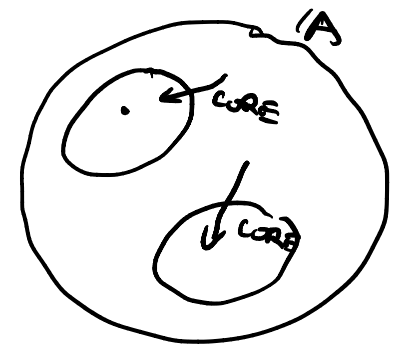


IN PARTICULAR ALL ON SCOREBOARD

PW $\neq$ ZARISKI:
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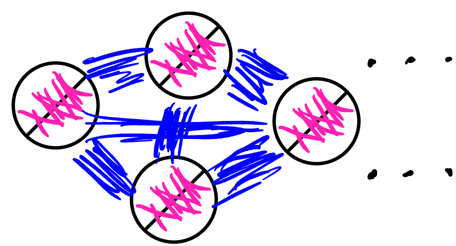

# EXCLUDING FINER TOPOLOGIES

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## DEFINITION

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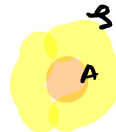


$\mathcal{S}$ HAS PROPERTY X WITH RESPECT TO $A$

$:\Longleftrightarrow$

$\forall s \in \mathcal{S} \ \exists f_s, g_s \in \mathcal{S} \ \exists \alpha_s \in A$ :

$$s = g_s \circ \alpha_s \circ f_s$$

AND

$\forall B$ NEIGHBOURHOOD OF $\alpha_s$ :

$g_s \circ (B \cap A) \circ f_s$ is NEIGHBOURHOOD OF $s$.

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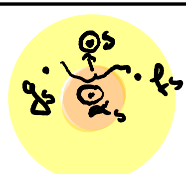

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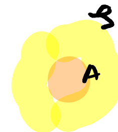


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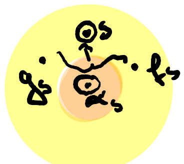


## THEOREM

(ELLIOTT + JONUŠAS + MESYAN + MITCHELL + MORAYNE + PÉRESSE '19)

$\mathcal{S}$ SEMIGROUP, POLISH TOPOLOGY $\mathcal{T}$ ON $\mathcal{S}$.

$A \leq \mathcal{S}$ SUBSEMIGROUP, PROPERTY $\mathsf{X}$.

- IF $A$ IS POLISH SUBGROUP
  $\Rightarrow \mathcal{T}$ IS A MAXIMAL POLISH TOPOLOGY ON $\mathcal{S}$.
- IF $A$ HAS AC
  $\Rightarrow \mathcal{S}$ HAS AC.

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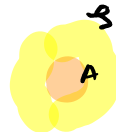


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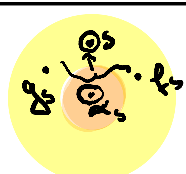

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## COROLLARY

$\mathrm{End}(\mathbb{A})$ HAS PROPERTY $\mathsf{X}$ WRT $\mathrm{Aut}(\mathbb{A})$

$\Rightarrow$ PW MAXIMAL POLISH TOPOLOGY.

**EXAMPLE** $\mathbb{A} = (\mathbb{N}, =)$ : $\mathrm{End}(\mathbb{A}) = \mathbb{N}^{\mathbb{N}}$, $\mathrm{Aut}(\mathbb{A}) = \mathrm{Sym}(\mathbb{N})$

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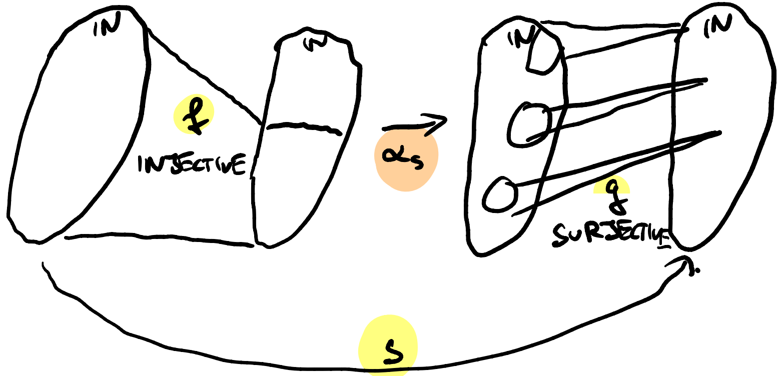
PROPERTY X :

$s \in \mathbb{N}^{\mathbb{N}}$




$s = g \circ \alpha_s \circ f$

SURJECTIVE INFINITE CLASSES

INJECTIVE CO-INFINITE RANGE

$\Rightarrow$ PW MAXIMAL POLISH TOPOLOGY ON $\mathbb{N}^{\mathbb{N}}$

**NON-EXAMPLE**

$\mathbb{A} = (\mathbb{Q}, \leq)$

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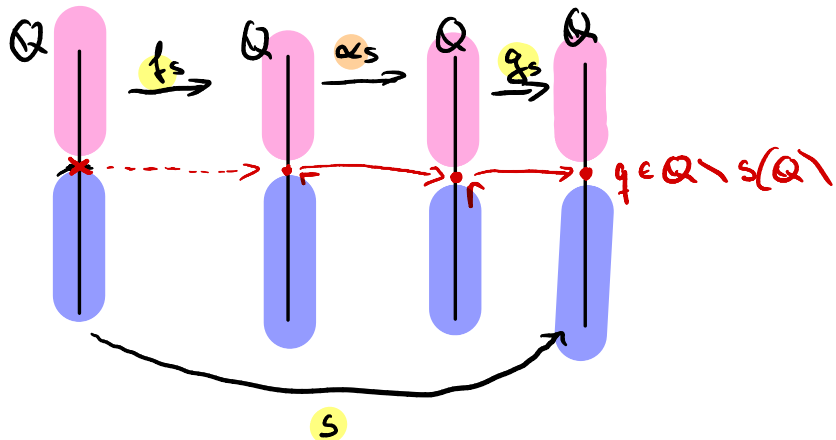
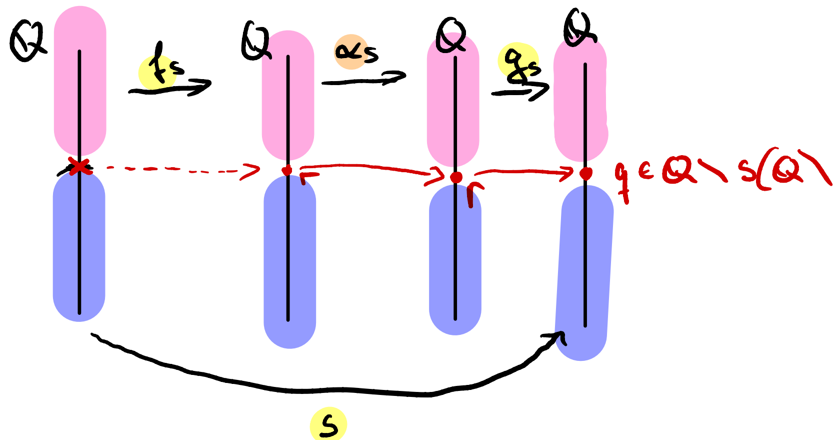



1. TAKE $s$ NON-SURJECTIVE, $q \notin s(Q)$

**NON-EXAMPLE** $\mathbb{A} = (\mathbb{Q}, \leq)$




1. TAKE $s$ NON-SURJECTIVE, $q \notin s(Q)$
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   - $\exists\, p \quad g_s(p) = q$
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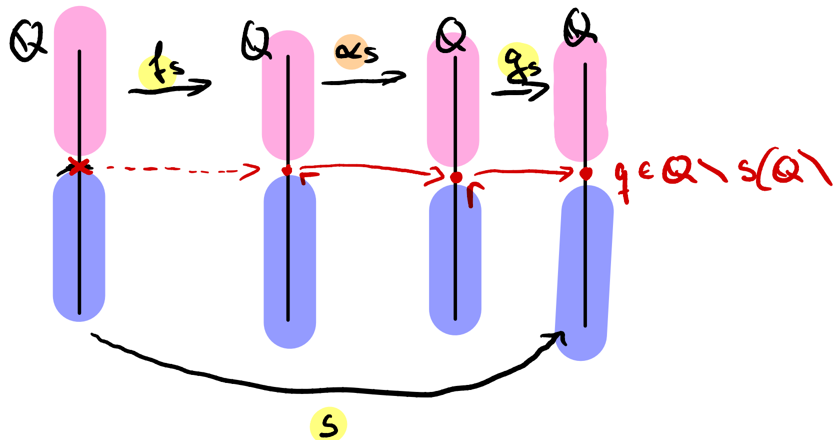
## NON-EXAMPLE

$\mathbb{A} = (Q, \leq)$

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3. IF $\tilde{\alpha}_s : r \mapsto p \Rightarrow g_s \circ \tilde{\alpha}_s \circ f_s$ MUST SEND BELOW $q$, ABOVE $q$

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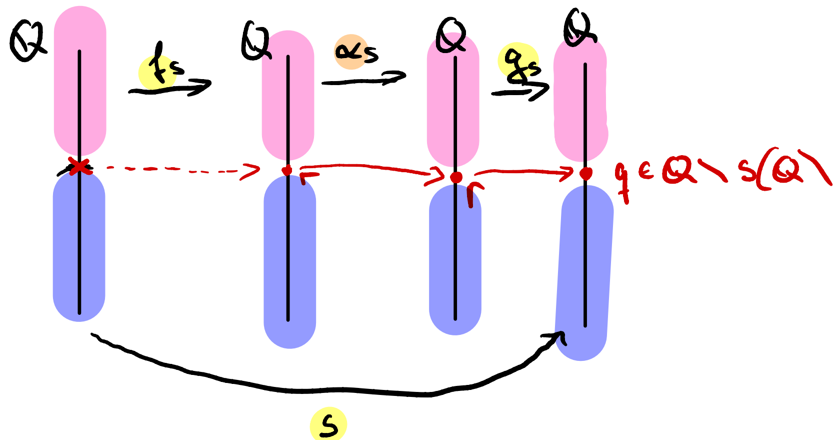


1. TAKE $s$ NON-SURJECTIVE, $q \notin s(\mathbb{Q})$

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   - $\exists \rho \quad g_s(\rho) = q$
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3. IF $\tilde{\alpha}_s : r \mapsto \rho \Rightarrow g_s \circ \tilde{\alpha}_s \circ f_s$ MUST SEND BELOW $q$, ABOVE $q$

4. NO NEIGHBOURHOOD OF $s$ IS THAT RESTRICTIVE!

SUFFICIENT CONDITIONS FOR PROPERTY X

## SUFFICIENT CONDITIONS FOR PROPERTY X

- $\mathbb{A}$ HOMOMORPHISM-HOMOGENEOUS $:\Longleftrightarrow$ EVERY FINITE PARTIAL ENDOMORPHISM EXTENDS TO GLOBAL ENDOMORPHISM

  EXAMPLE: RANDOM GRAPH

  NON-EXAMPLE: $\triangle$-FREE GRAPH

# SUFFICIENT CONDITIONS FOR PROPERTY X

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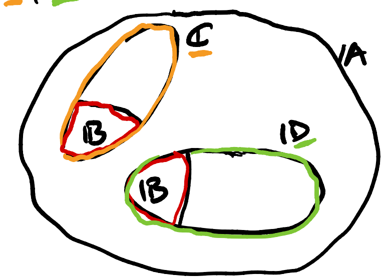
  EXAMPLE: RANDOM GRAPH

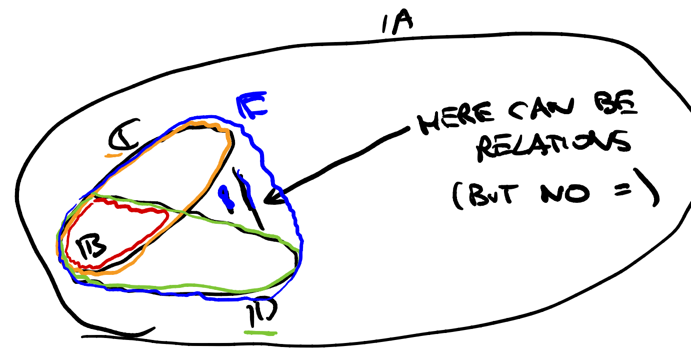
  NON-EXAMPLE: $\triangle$-FREE GRAPH

- $\mathbb{A}$ HAS STRONG AMALGAMATION WITH HOMOMORPHISM GLUING $:\Longleftrightarrow$

$\forall \mathbb{B}, \mathbb{C}, \mathbb{D}$ FINITE IN $\mathbb{A}$




$\exists \mathbb{E}$ IN $\mathbb{A}$ :



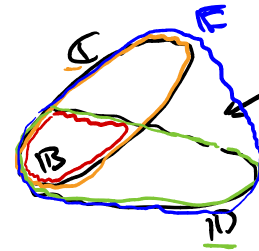

SUCH THAT $\forall\ g: \mathbb{E} \to \mathbb{F}$,

$g|_{\mathbb{C}}$ HOMO, $g|_{\mathbb{D}}$ HOMO

$\Rightarrow g$ HOMO

> "$\mathbb{E}$ IS STRONG AMALGAM WITH SMALLEST RELATIONS"

## EXAMPLES




- FREE AMALGAMATION $\Rightarrow$ SAP + HG
- IN PARTICULAR: RANDOM GRAPH, DIGRAPH, ... HAVE SAP + HG
- RANDOM POSET $(\mathbb{P}, \leq)$, $(\mathbb{P}, <)$ HAVE SAP + HG
- EQUIVALENCE REL. $\mathbb{E}$ HAS SAP + HG
- $(\mathbb{Q}, \leq)$, $(\mathbb{Q}, <)$, $\mathbb{T}$ RANDOM TOURNAMENT: NO HG!

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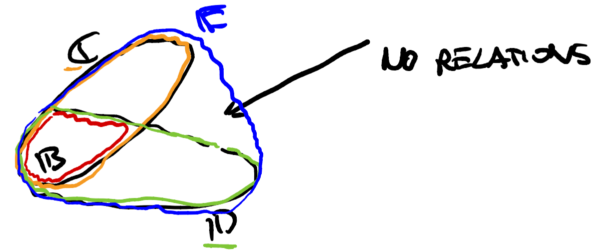


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**THEOREM** (ELLIOTT + JONUŠAS + MITCHELL + PÉRESSE + P. '21)

LET $\mathbb{A}$ BE HOMOGENEOUS, HOMOMORPHISM-HOMOGENEOUS, SAP + HG.

$\Rightarrow$ $\mathrm{End}(\mathbb{A})$ HAS PROPERTY **X** WRT $\mathrm{Aut}(\mathbb{A})$,

PW IS A MAXIMAL POLISH TOPOLOGY.

## COROLLARY

UNIQUE POLISH TOP:
End / Pol OF

- RANDOM GRAPH
- DIGRAPH
- EQUIVALENCE
- POSET ≤
- POSET <
- EMPTY STRUCTURE

AUTOMATIC CONTINUITY:
End / Pol OF

- RANDOM GRAPH
- DIGRAPH
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**THEOREM** (P. + SCHINDLER '23)

End $(\mathbb{Q}, \leq)$ HAS UNIQUE POLISH TOPOLOGY.

← NEEDS BAIRE CATEGORY THM

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End $(\mathbb{Q}, \leq)$ HAS UNIQUE POLISH TOPOLOGY.

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**OPEN PROBLEMS**

- THE DUAL OF THE RANDOM EQUIVALENCE RELATION?
- IS PW ALWAYS THE COARSEST $T_2$ TOPOLOGY?

# Thank you!

| | UNIQUE POLISH (UP) | AUTOMATIC CONT. (AC) | ACTION REC. (AR) | AUTOMATIC HOMEO (AH) |
|---|---|---|---|---|
| Aut | $(\mathbb{N},=)$ Gaughan '70s<br>$(\mathbb{Q},<)$ Rosendal + Solecki '07<br>$\mathbb{G}$ Kechris + Rosendal '07 | $(\mathbb{N},=)$ Dixon, Neumann, Thomas '86, Semmes '81<br>$(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89<br>$\mathbb{G}$, ω-stable Hodges + Hodkinson + Lascar + Shelah '90s<br>$\mathbb{H}_n$ Herwig '90s | $(\mathbb{Q},<)$, $\mathbb{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94<br>Hypergraphs, Henson digraphs Barbina + Macpherson '07 | $(\mathbb{Q},<)$, $\mathbb{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 |
| End | $\mathbb{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{Q},\leq)$ Schneider + P. '23<br>$(\mathbb{N},=)$ EJMMMP '20 | $\mathbb{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{N},=)$ EJMMMP '20 | $(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathbb{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19 | $(\mathbb{N},\neq)$, $\mathbb{G}$, $\mathbb{T}$ Bodirsky + P. + Pongrácz '13<br>$(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17 |
| Pol | $\mathbb{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{N},=)$ Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '20 | $\mathbb{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21<br>$(\mathbb{N},=)$ EJMMMP '20 | $(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathbb{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19 | $\supseteq \mathbb{N}^{\mathbb{N}}$, $\mathbb{G}$ Bodirsky + P. + Pongrácz '13<br>$(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Pech + Pech '18<br>$(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17<br>$(\mathbb{Q},<)$ Behrisch + Vargas-García '19 |

$\mathbb{G}$ ... RANDOM GRAPH
$\mathbb{P}$ ... RANDOM POSET
$\mathbb{T}$ ... RANDOM TOURNAMENT
$\mathbb{D}$ ... RANDOM DIGRAPH
$\mathbb{E}$ ... RANDOM EQUIVALENCE RELATION
$\mathbb{H}_n$ ... RANDOM $K_n$-FREE GRAPH
$\mathbb{B}$ ... COUNTABLE ATOMLESS BOOLEAN ALG
ω-CAT
NO ALGEBRAICITY

40!

FUNDED BY THE EUROPEAN UNION (ERC, POCOCOP, 101071674).
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