TOPOLOGIES ON ENDOMORPHISM MONOIDS OF GENERIC STRUCTURES

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PART I : MOTIVATION & RESULTS

- ALGEBRAIC-TOPOLOGICAL STRUCTURES
- AUTOMORPHISM GROUPS, ENDOMORPHISM MONOIDS, POLYMORPHISM CLONES
- RECONSTRUCTION NOTIONS
- THE SCOREBOARD

PART II : METHODS

- THE ZARISKI TOPOLOGY
- HOMOMORPHISM GLUING & HOMOGENEITY

MANY MATHEMATICAL OBJECTS CARRY:

- ALGEBRAIC STRUCTURE
- TOPOLOGICAL STRUCTURE

} COMPATIBLE: ALGEBRAIC OPERATIONS CONTINUOUS.

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- ALGEBRA $(A; (f_i)_{i\in I})$: COMPOSITION + POINTWISE CONVERGENCE.

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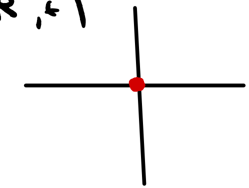

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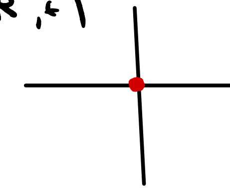

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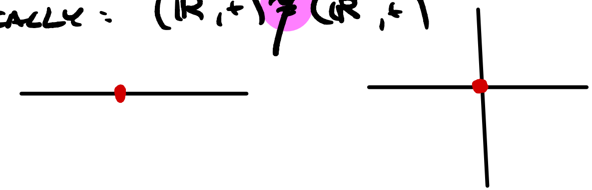

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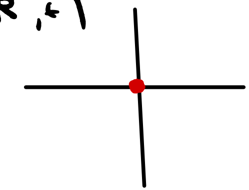

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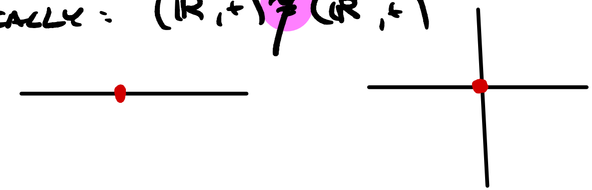

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QUESTION IS THERE A UNIQUE ... TOPOLOGY?

... E.G.
- T_2 / HAUSDORFF
- SEPARABLE
- POLISH = SEPARABLE + COMPLETELY METRIZABLE

GROUPS

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ANSWER NO:

- EVERY T_1 TOPOLOGY $\supseteq$ PW (GAUGHAN '70s)
- $\mathcal{J} \subseteq \mathcal{J}'$, BOTH POLISH $\Rightarrow$ $\mathcal{J} = \mathcal{J}'$

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UNIQUE POLISH TOPOLOGY:

- ISOMETRY GROUP OF THE URYSOHN SPACE / SPHERE (SABOK '13)
- HOMEOMORPHISM GROUPS OF $[0,1]^{\mathbb{N}}$, $2^{\mathbb{N}}$ (KALLMAN '80s)
- $\mathrm{Aut}(\mathbb{Q},<)$ (HODGES, HODKINSON, LASCAR, SHELAH '93 / SOLECKI + ROSENDAL '07)
- $\mathrm{Aut}(G)$ G... RANDOM GRAPH (HRUSHOVSKI '92 / KECHRIS + ROSENDAL '07)

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- $\mathbb{N}^{\mathbb{N}} = \mathrm{End}(\mathbb{N}, =)$
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(ELLIOTT + JONUŠAS + MESYAN + MITCHELL + MORAYNE + PÉRESSE '19)

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$M_n := \{f \in \mathbb{N}^{\mathbb{N}} \mid |\mathbb{N} \setminus f(\mathbb{N})| = n\}$ $(n \leq \omega)$

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INBETWEEN: RESTRICT n

(NO CLASSIFICATION)

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TOPOLOGY IS IRRELEVANT (IN A DICHOTOMY CONJECTURE FOR INFINITE DOMAIN CONSTRAINT SATISFACTION PROBLEMS)*

LIBOR BARTO† AND MICHAEL PINSKER‡

WHAT FOR?

INSIGHT INTO RELATIONSHIP BETWEEN ALGEBRAIC & TOPOLOGICAL STRUCTURE:
IS TOPOLOGY RELEVANT (OR ONLY CONVENIENT)?

PRACTICAL USE ALGEBRAIC OBJECT AS INVARIANT

What for?

Insight into relationship between algebraic & topological structure: is topology relevant (or only convenient)?

Practical use algebraic object as invariant

Examples Let $\mathbb{A}, \mathbb{B}$ be countable <u>ω-categorical</u>

- $\mathbb{A}, \mathbb{B}$ first-order bi-interpretable $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ (Ahlbrandt Ziegler '80s)
  if $\mathbb{A}$ has unique Polish top.: $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ Coquand '81
- $\mathbb{A}, \mathbb{B}$ existential-positive bi-int. $\Leftrightarrow$ $\mathrm{End}(\mathbb{A}) \cong \mathrm{End}(\mathbb{B})$ (Bodirsky + Junker '08)
- $\mathbb{A}, \mathbb{B}$ primitive-positive bi-int. $\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \cong \mathrm{Pol}(\mathbb{B})$

(Bodirsky + P. '11)

VARIANT AUTOMATIC CONTINUITY AC

Aut($\mathbb{A}$) HAS AC :⟺ $\forall \varphi$: Aut($\mathbb{A}$) → Sym($\mathbb{N}$) : φ CONTINUOUS

End($\mathbb{A}$) HAS AC :⟺ End($\mathbb{A}$) → $\mathbb{N}^{\mathbb{N}}$

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MEANING FOR ω-CATEGORICAL $\mathbb{A}$:

$\mathbb{B}$ HAS FO-INTERPRETATION IN $\mathbb{A}$ $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$ (MORE OR LESS)

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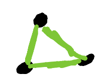

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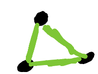
EXAMPLE CAN $\mathbb{A}$ PP-INTERPRET THE GRAPH K_3? 

THEOREM $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}($$)$ $\Leftrightarrow$ $\exists e, f, s \in \mathrm{Pol}(\mathbb{A})$:

$\forall x,y,z \quad e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$

(BARTO + P. '16, NOT TRUE AS STATED)

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$

$\mathbb{A}$... COUNTABLE ω-CATEGORICAL

THREE NOTIONS:

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THREE NOTIONS:

HARD TO FAIL (over $\mathrm{Aut}(\mathbb{A})$) — FAILS EASILY (over $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$)

UP: PW IS UNIQUE POLISH TOPOLOGY ON $\mathrm{Aut}(\mathbb{A})$ / $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$

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HARD TO FAIL (over $\mathrm{Aut}(\mathbb{A})$); FAILS EASILY (over $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$)

UP: PW IS UNIQUE POLISH TOPOLOGY ON $\mathrm{Aut}(\mathbb{A})$ / $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$

AC: EVERY $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ } HARD TO FAIL

IS CONTINUOUS

FAILS EASILY {
$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$
$\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$

(AUTOMATIC CONTINUITY)

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$
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THREE NOTIONS:

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$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

$\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$

(FAILS EASILY for End, Pol)

IS CONTINUOUS

(AUTOMATIC CONTINUITY)

AH: EVERY ISOMORPHISM $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$

$\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$

$\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

IS A HOMEOMORPHISM

(AUTOMATIC HOMEOMORPHICITY)

HARD TO FAIL

EVANS + HEWITT '90 + BODIRSKY + KOMPATSCHER + P. '15

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$
$\mathbb{A}$... COUNTABLE ω-CATEGORICAL

THREE NOTIONS:

UP : PW IS UNIQUE POLISH TOPOLOGY ON $\mathrm{Aut}(\mathbb{A})$ / $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$

(HARD TO FAIL for $\mathrm{Aut}(\mathbb{A})$; FAILS EASILY for $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$)

AC : EVERY $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ } HARD TO FAIL
$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$
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IS CONTINUOUS

(AUTOMATIC CONTINUITY)

AH : EVERY ISOMORPHISM $\varphi: \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$
$\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$
$\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$
IS A HOMEOMORPHISM

HARD TO FAIL

(AUTOMATIC HOMEOMORPHICITY)

EVANS + HEWITT '90
+ BODIRSKY + KOMPATSCHER + P. '15

AR : EVERY ISOMORPHISM φ IS INDUCED BY BIJECTION $f: \mathbb{A} \to \mathbb{B}$

(ACTION RECONSTRUCTION)

UP $\longrightarrow$ AH $\leftarrow$ AC
AH $\leftarrow$ AR

FOR GROUPS (OPEN FOR MONOIDS)

AR USUALLY RESTRICTED TO $\mathbb{B}$ SATISFYING CONDITIONS : NO ALGEBRAICITY

THE SCOREBOARD

(INCOMPLETE + PROBABLY INCORRECT)

	UNIQUE POLISH (UP)	AUTOMATIC CONT. (AC)	ACTION REC. (AR)	AUTOMATIC HOMEO (AH)
Aut	$(\mathbb{N},=)$ Gaughan '70s $(\mathbb{Q},<)$ Rosendal + Solecki '07 $\mathcal{G}$ Kechris + Rosendal '07	$(\mathbb{N},=)$ Dixon, Neumann, Thomas '86 Semmes '81 $(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89 $\mathcal{G}$, ω-stable Hodges + Hodkinson + Lascar + Shelah '80s $\mathbb{H}_n$ Herwig '80s	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 Hypergraphs, Henson digraphs Barbina, Macpherson '07	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94
End	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20 $(\mathbb{Q},\leq)$ Schneider + P. '23	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19	$(\mathbb{N},\neq)$, $\mathcal{G}$, $\mathbb{T}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17
Pol	$\mathcal{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '20	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19	$\supseteq \mathbb{N}^{\mathbb{N}}$, $\mathcal{G}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Pech + Pech '18 $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17 $(\mathbb{Q},<)$ Behrisch + Vargas-García '19

$\mathcal{G}$ ... RANDOM GRAPH

$\mathbb{P}$ ... RANDOM POSET

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$\mathbb{H}_n$ ... RANDOM K_n-FREE GRAPH

$\mathbb{B}$ ... COUNTABLE ATOMLESS BOOLEAN ALG

ω-CAT

NO ALGEBRAICITY

PART I : MOTIVATION & RESULTS

- ALGEBRAIC-TOPOLOGICAL STRUCTURES
- AUTOMORPHISM GROUPS, ENDOMORPHISM MONOIDS, POLYMORPHISM CLONES
- RECONSTRUCTION NOTIONS
- THE SCOREBOARD

PART II : METHODS

- THE ZARISKI TOPOLOGY
- HOMOMORPHISM GLUING & HOMOGENEITY

ZARISKI TOPOLOGY

ZARISKI TOPOLOGY

- CLOSED SETS: ALGEBRAICALLY GIVEN BY $\{ s \mid a_1 \circ s \circ a_2 \circ \dots \circ a_n \circ s = b_1 \circ s \circ b_2 \circ \dots \circ b_m \circ s \}$ (AS OPPOSED TO PW)
- ANY T_1 SEMIGROUP TOPOLOGY $\supseteq$ ZARISKI
- ZARISKI NEED NOT BE T_2 OR A SEMIGROUP TOPOLOGY!

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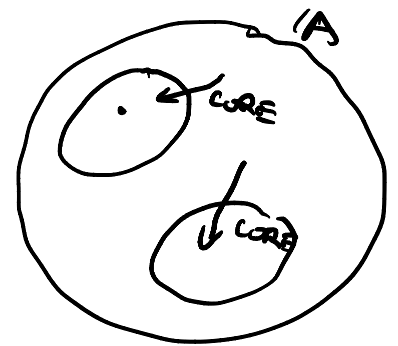
THEOREM (P. + SCHINDLER '23)

$\mathbb{A}$ ω-CATEGORICAL, NO ALGEBRAICITY

- EVERY ELEMENT CONTAINED IN CORE
- CORE FINITE OR NO ALGEBRAICITY

$\Rightarrow$ PW = ZARISKI ON $\mathrm{End}(\mathbb{A})$



IN PARTICULAR ALL ON SCOREBOARD

ZARISKI TOPOLOGY

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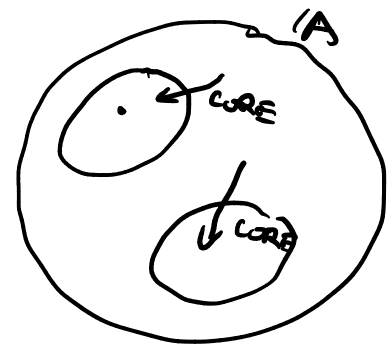
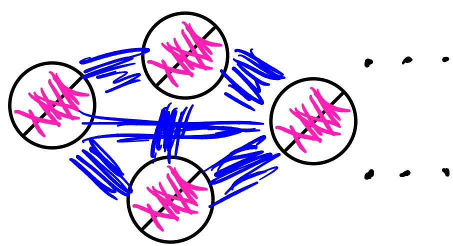


IN PARTICULAR ALL ON SCOREBOARD

PW ≠ ZARISKI:
(P. + SCHINDLER '23)



EXCLUDING FINER TOPOLOGIES

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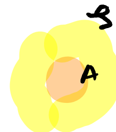
DEFINITION



$\mathcal{S}$ topological semigroup,

$A \subseteq \mathcal{S}$ subsemigroup.

$\mathcal{S}$ has property X with respect to A

$:\iff$

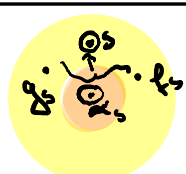
$\forall s \in \mathcal{S} \; \exists f_s, g_s \in \mathcal{S} \; \exists \alpha_s \in A$:

$$s = g_s \circ \alpha_s \circ f_s$$

and

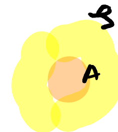
$\forall B$ neighbourhood of α_s :

$g_s \circ (B \cap A) \circ f_s$ is neighbourhood of s.



Excluding finer topologies

Definition



S topological semigroup,

$A \subseteq S$ subsemigroup.

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$:\Leftrightarrow$

$\forall s \in S \ \exists f_s, g_s \in S \ \exists \alpha_s \in A$:

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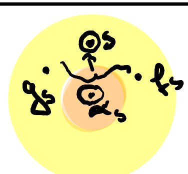


Theorem

(Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '19)

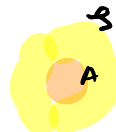
S semigroup, Polish topology $\mathcal{T}$ on S.

$A \leq S$ subsemigroup, property X.

- If A is Polish subgroup $\Rightarrow$ $\mathcal{T}$ is a maximal Polish topology on S.
- If A has AC $\Rightarrow$ S has AC.

EXCLUDING FINER TOPOLOGIES

DEFINITION



S TOPOLOGICAL SEMIGROUP,
$A \leq S$ SUBSEMIGROUP.

S HAS PROPERTY **X** WITH RESPECT TO A

$:\Longleftrightarrow$

$\forall s \in S \ \exists f_s, g_s \in S \ \exists \alpha_s \in A:$

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AND

$\forall B$ NEIGHBOURHOOD OF α_s:

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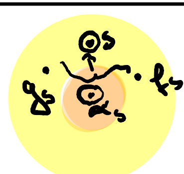


THEOREM

(ELLIOTT + JONUŠAS + MESYAN + MITCHELL + MORAYNE + PÉRESSE '19)

S SEMIGROUP, POLISH TOPOLOGY $\mathcal{T}$ ON S.

$A \leq S$ SUBSEMIGROUP, PROPERTY **X**.

- IF A is POLISH SUBGROUP
  $\Longrightarrow$ $\mathcal{T}$ is A MAXIMAL POLISH TOPOLOGY ON S.
- IF A HAS AC
  $\Longrightarrow$ S HAS AC.

COROLLARY

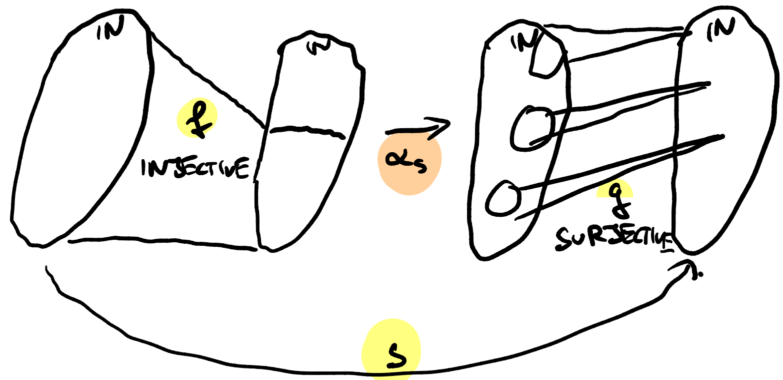
$\mathrm{End}(\mathbb{A})$ HAS PROPERTY **X** WRT $\mathrm{Aut}(\mathbb{A})$
$\Longrightarrow$ PW MAXIMAL POLISH TOPOLOGY.

EXAMPLE $\mathbb{A} = (\mathbb{N}, =)$: $\mathrm{End}(\mathbb{A}) = \mathbb{N}^{\mathbb{N}}$, $\mathrm{Aut}(\mathbb{A}) = \mathrm{Sym}(\mathbb{N})$

EXAMPLE $\mathbb{A} = (\mathbb{N}, =)$: $\mathrm{End}(\mathbb{A}) = \mathbb{N}^{\mathbb{N}}$, $\mathrm{Aut}(\mathbb{A}) = \mathrm{Sym}(\mathbb{N})$

PROPERTY **X** :

$s \in \mathbb{N}^{\mathbb{N}}$



$s = g \circ \alpha_s \circ f$

SURJECTIVE INFINITE CLASSES (g)

INJECTIVE CO-INFINITE RANGE (f)

$\Rightarrow$ PW MAXIMAL POLISH TOPOLOGY ON $\mathbb{N}^{\mathbb{N}}$

NON-EXAMPLE $\mathbb{A} = (\mathbb{Q}, \leq)$

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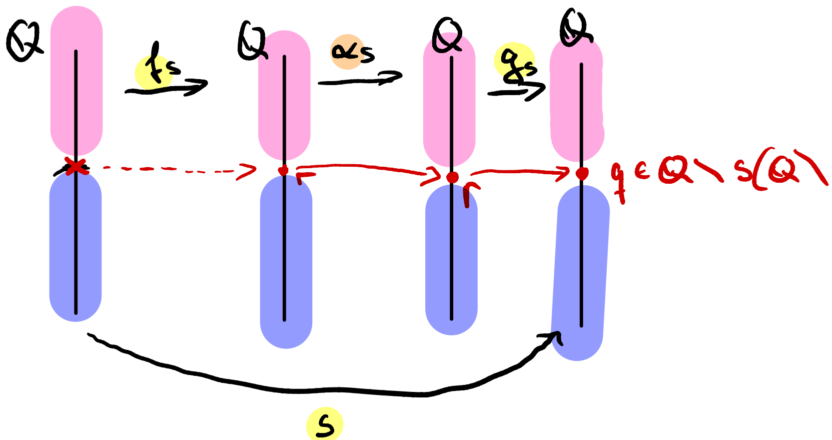
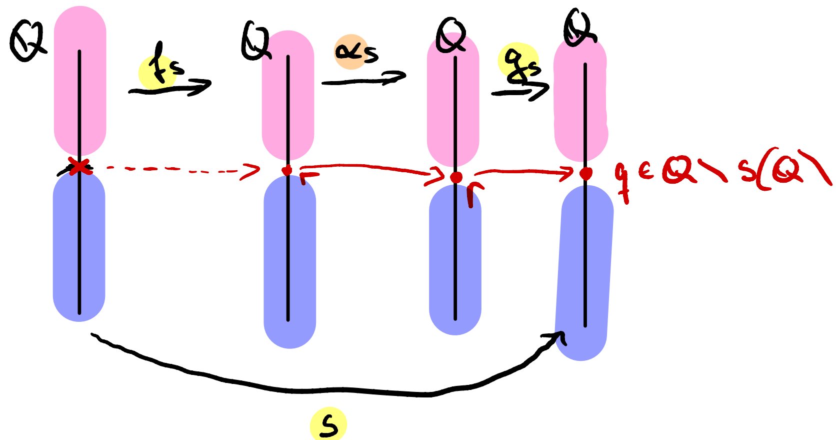


1. TAKE s NON-SURJECTIVE, $q \notin s(\mathbb{Q})$

NON-EXAMPLE

$\mathbb{A} = (Q, \leq)$



1. TAKE s NON-SURJECTIVE, $q \notin s(Q)$
2. (!) g_s HAS TO BE SURJECTIVE $\Rightarrow$
   - $\exists\, p \quad g_s(p) = q$
   - $\exists\, r \quad \alpha_s(r) = p$
   - $r \notin f_s(Q)$

NON-EXAMPLE

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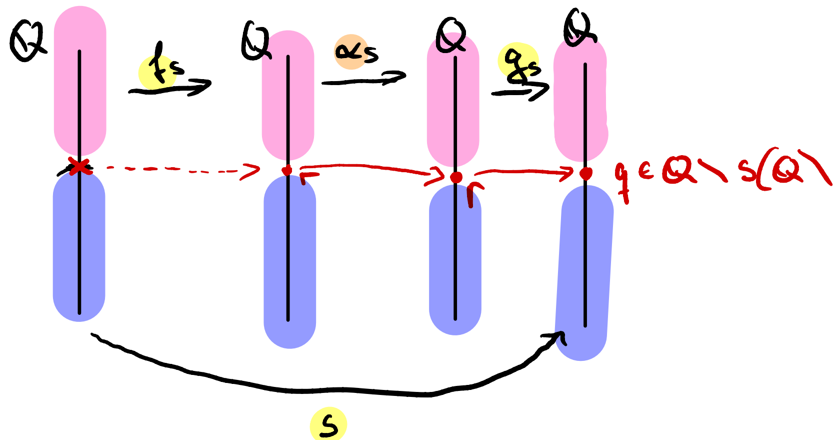

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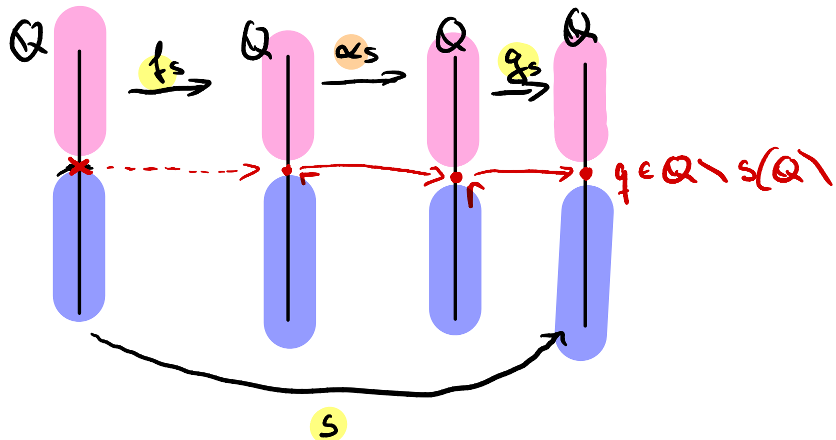
3. IF $\tilde{\alpha}_s : r \mapsto p \Rightarrow g_s \circ \tilde{\alpha}_s \circ f_s$ MUST SEND (blue) BELOW q, (pink) ABOVE q

NON-EXAMPLE

$\mathbb{A} = (\mathbb{Q}, \leq)$



1. TAKE s NON-SURJECTIVE, $q \notin s(\mathbb{Q})$

2. g_s HAS TO BE SURJECTIVE $\Rightarrow$
   - $\exists \rho \quad g_s(\rho) = q$
   - $\exists r \quad \alpha_s(r) = \rho$
   - $r \notin f_s(\mathbb{Q})$

3. IF $\tilde{\alpha}_s : r \mapsto \rho \Rightarrow g_s \circ \tilde{\alpha}_s \circ f_s$ MUST SEND (blue) BELOW q, (pink) ABOVE q

4. NO NEIGHBOURHOOD OF s IS THAT RESTRICTIVE!

SUFFICIENT CONDITIONS FOR PROPERTY X

,

SUFFICIENT CONDITIONS FOR PROPERTY X

- $\mathbb{A}$ **HOMOMORPHISM-HOMOGENEOUS** $:\Longleftrightarrow$ EVERY FINITE PARTIAL ENDOMORPHISM EXTENDS TO GLOBAL ENDOMORPHISM

  EXAMPLE: RANDOM GRAPH

  NON-EXAMPLE: $\triangle$-FREE GRAPH

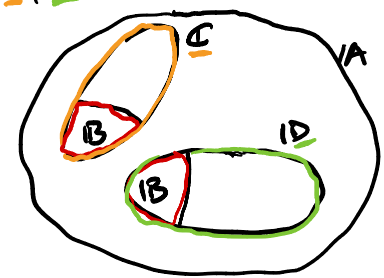
SUFFICIENT CONDITIONS FOR PROPERTY X

- $\mathbb{A}$ **HOMOMORPHISM-HOMOGENEOUS** :⟺ EVERY FINITE PARTIAL ENDOMORPHISM EXTENDS TO GLOBAL ENDOMORPHISM

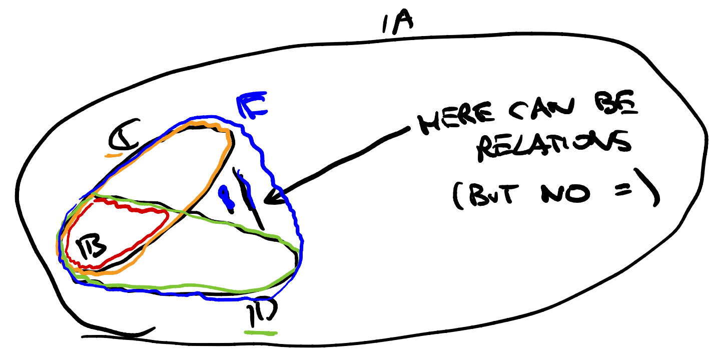
  EXAMPLE: RANDOM GRAPH
  NON-EXAMPLE: △-FREE GRAPH

- $\mathbb{A}$ HAS **STRONG AMALGAMATION WITH HOMOMORPHISM GLUING** :⟺

  $\forall \mathbb{B}, \mathbb{C}, \mathbb{D}$ FINITE IN $\mathbb{A}$



$\exists \mathbb{E}$ IN $\mathbb{A}$:



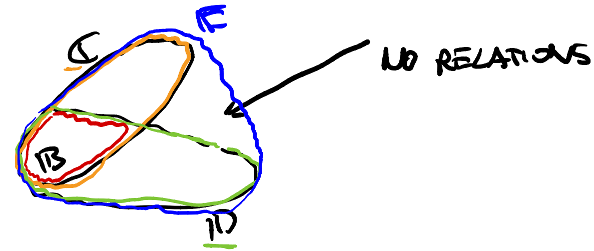
SUCH THAT $\forall\ g: \mathbb{E} \to \mathbb{F}$,

$g|_{\mathbb{C}}$ HOMO, $g|_{\mathbb{D}}$ HOMO

$\Rightarrow g$ HOMO

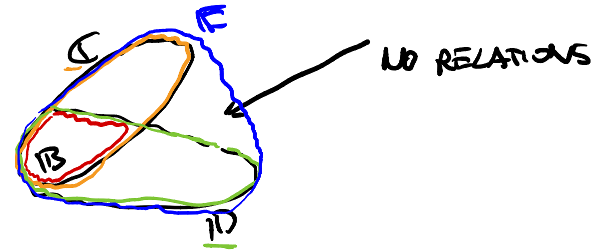
> "$\mathbb{E}$ IS STRONG AMALGAM WITH SMALLEST RELATIONS"

EXAMPLES



- FREE AMALGAMATION $\Rightarrow$ SAP + HG
- IN PARTICULAR: RANDOM GRAPH, DIGRAPH, ... HAVE SAP + HG
- RANDOM POSET $(\mathbb{P}, \leq)$, $(\mathbb{P}, <)$ HAVE SAP + HG
- EQUIVALENCE REL. $\mathbb{E}$ HAS SAP + HG
- $(\mathbb{Q}, \leq)$, $(\mathbb{Q}, <)$, $\mathbb{T}$ RANDOM TOURNAMENT: NO HG!

EXAMPLES



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THEOREM (ELLIOTT + JONUŠAS + MITCHELL + PÉRESSE + P. '21)

LET $\mathbb{A}$ BE HOMOGENEOUS, HOMOMORPHISM-HOMOGENEOUS, SAP + HG.

⇒ End($\mathbb{A}$) HAS PROPERTY X WRT Aut($\mathbb{A}$),

PW IS A MAXIMAL POLISH TOPOLOGY.

COROLLARY

UNIQUE POLISH TOP:
End / Pol OF

- RANDOM GRAPH
- DIGRAPH
- EQUIVALENCE
- POSET ≤
- POSET <
- EMPTY STRUCTURE

AUTOMATIC CONTINUITY:
End / Pol OF

- RANDOM GRAPH
- DIGRAPH
- EQUIVALENCE
- EMPTY STRUCTURE

COROLLARY

UNIQUE POLISH TOP: End / Pol OF

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- DIGRAPH
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- DIGRAPH
- EQUIVALENCE
- EMPTY STRUCTURE

THEOREM (P. + SCHINDLER '23)

End $(\mathbb{Q}, \leq)$ HAS UNIQUE POLISH TOPOLOGY.

← NEEDS BAIRE CATEGORY THM

COROLLARY

UNIQUE POLISH TOP:
End / Pol OF

- RANDOM GRAPH
- DIGRAPH
- EQUIVALENCE
- POSET $\leq$
- POSET $<$
- EMPTY STRUCTURE

AUTOMATIC CONTINUITY:
End / Pol OF

- RANDOM GRAPH
- DIGRAPH
- EQUIVALENCE
- EMPTY STRUCTURE

THEOREM (P. + SCHINDLER '23)

End $(\mathbb{Q}, \leq)$ HAS UNIQUE POLISH TOPOLOGY.

← NEEDS BAIRE CATEGORY THM

OPEN PROBLEMS

- THE DUAL OF THE RANDOM EQUIVALENCE RELATION?
- IS PW ALWAYS THE COARSEST T_2 TOPOLOGY?

Thank you!

	Unique Polish (UP)	Automatic Cont. (AC)	Action Rec. (AR)	Automatic Homeo (AH)
Aut	$(\mathbb{N},=)$ Gaughan '70s $(\mathbb{Q},<)$ Rosendal + Solecki '07 $\mathcal{G}$ Kechris + Rosendal '07	$(\mathbb{N},=)$ Dixon, Neumann, Thomas '86, Semmes '81 $(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89 $\mathcal{G}$, ω-stable Hodges + Hodkinson + Lascar + Shelah '80s $\mathbb{H}_n$ Herwig '80s	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 Hypergraphs, Henson digraphs Barbina + Macpherson '07	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94
End	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{Q},\leq)$ Schindler + P. '23 $(\mathbb{N},=)$ EJMMMP '20	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19	$(\mathbb{N},+)$, $\mathcal{G}$, $\mathbb{T}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17
Pol	$\mathcal{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ Elliott + Jonušas + Mesyan + Mitchell + Morayne + Péresse '20	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-García '19	$\supseteq \mathbb{N}^{\mathbb{N}}$, $\mathcal{G}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Pech + Pech '18 $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-García '17 $(\mathbb{Q},<)$ Behrisch + Vargas-García '19

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ω-CAT
NO ALGEBRAICITY

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