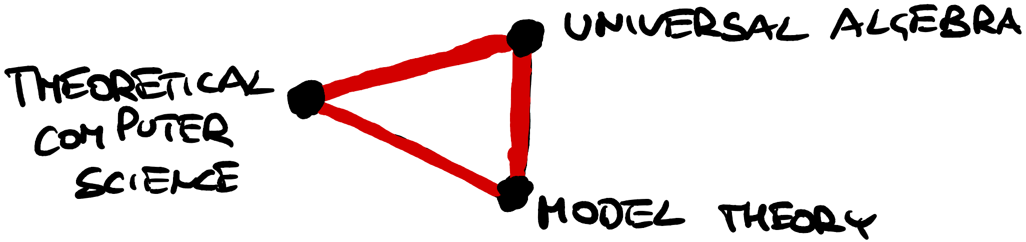
THE POWER OF POLYMORPHISMS



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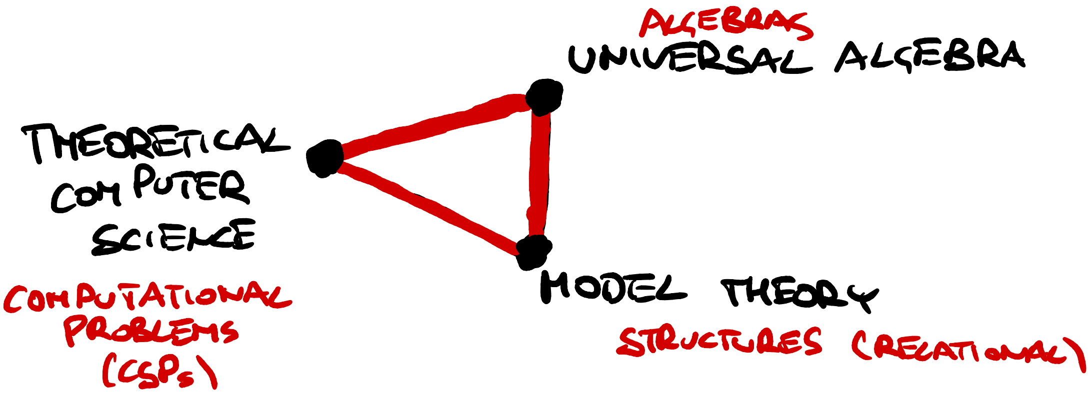

PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.
ALGEBRAS
RELATIONAL STRUCTURES

PART I : THE MATHEMATICS OF FINITE-DOMAIN CSPs

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \cdots x_n$ SATISFYING ALL CONSTRAINTS ?

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- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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- SUDOKU
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- SOLVING EQUATIONS

MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
  E.G. $\{0,1\}, \{0,1,2\}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \dots$
- ALLOWED CONSTRAINTS (FIXED)
  $C_1, \dots, C_m$ RELATIONS ON V
  $C_i \subseteq V^{a_i}$

SO $\mathbb{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE
= "TEMPLATE"

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS C_{i_1}(VARIABLES)
  $\vdots$
  C_{i_ℓ}(VARIABLES)

QUESTION ∃ SOLUTION $x_i \mapsto a_i \in \mathbb{A}$?

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QUESTION $\exists$ SOLUTION $x_i \mapsto a_i \in A$?

ALTERNATIVE: GIVEN **PP-SENTENCE**

$\varphi \equiv \exists x_1 \dots \exists x_n \, C_{i_1}(\text{VAR}) \wedge \dots \wedge C_{i_\ell}(\text{VAR})$

QUESTION: $\mathbb{A} \models \varphi$?

PP... PRIMITIVE POSITIVE

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PP... PRIMITIVE POSITIVE

META-QUESTION

FOR WHAT $\mathbb{A}$ IS CSP($\mathbb{A}$) EASY / HARD ?

P : $\exists$ ALGORITHM PROVIDING ANSWER IN p(# VARIABLES) STEPS, p POLYNOMIAL

$\neq$

NP: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLE

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, \bar{+}, \bar{\cdot})$

CSP($\mathbb{A}$)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS

$x_1 \cdot x_1 = x_2$

$x_2 + x_1 = x_3$

$x_3 = 1$

QUESTION

- SOLUTION IN $\mathbb{Z}$?

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UNDECIDABLE (MATYIASEVIC '77)

HILBERT'S 10TH PROBLEM

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- SAME OVER $\mathbb{Z}_p$

NP-COMPLETE
↳ IF IN P ⇒ P = NP

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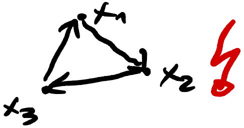
IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$
  $x_2 < x_3$
  $x_3 < x_1$
  $\vdots$

QUESTION
- SOLUTION IN $\mathbb{Q}$?



IN P

EXAMPLE

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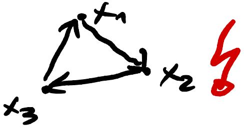
IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

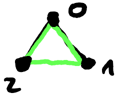
CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$
  $x_2 < x_3$
  $x_3 < x_1$
  ⋮

QUESTION
- SOLUTION IN $\mathbb{Q}$?



IN P

- $\mathbb{A} = K_3 =$  $= (\{0,1,2\}, E)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $E(x_1, x_2)$
  $E(x_2, x_3)$
  ⋮

QUESTION
- SOLUTION IN $\{0,1,2\}$?

3-COLORING PROBLEM = NP-COMPLETE

ALTERNATIVE FORMULATION OF CSP($\mathbb{A}$):

GIVEN: FINITE $\mathbb{B}$

QUESTION $\mathbb{B} \to \mathbb{A}$?, i.E.

$\exists h : \mathbb{B} \to \mathbb{A}$ HOMOMORPHISM?

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"SAME" PROBLEM:

- PP-SENTENCE $\varphi = \exists x_1 \dots x_n \, C_{i_1}(\ ) \wedge \dots \wedge C_{i_\ell}(\ )$

$\rightsquigarrow$ STRUCTURE: DOMAIN $\{x_1 \dots x_n\}$

$\mathbb{B}_\varphi$ RELATIONS C_i GIVEN BY CONSTRAINTS

"CANONICAL DATABASE"

THEN $\mathbb{B}_\varphi \to \mathbb{A} \Leftrightarrow \mathbb{A} \models \varphi$

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  ⤳ $\varphi_{\mathbb{B}} = \exists b_1 \dots b_\ell \bigwedge C_i(\;)$

  "CANONICAL QUERY OF $\mathbb{B}$"

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EXAMPLE $\mathbb{A} = K_2 =$

GIVEN DIGRAPH $\mathbb{B}$

QUESTION $\mathbb{B} \to K_2$?

2-COLORING PROBLEM

EXERCISE CSP$(\mathbb{Q}, <) = ?$

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EXERCISE $\mathrm{CSP}(\mathbb{Q}, <) = ?$

COMPUTATIONAL COMPLEXITY OF CSP(A) CAN BE:

- UNDECIDABLE
- IN P
- NP-COMPLETE
- ANYTHING:
  EVERY COMPUTATIONAL PROBLEM IS POLYNOMIAL-TIME TURING-EQUIVALENT TO SOME CSP ($\mathbb{A}$)

$\mathbb{A}$ FINITE $\Rightarrow$ CSP($\mathbb{A}$) IN NP

THEOREM (BULATOV, ZHUK '17)
(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P OR

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(P $\neq$ NP)

CSP($\mathbb{A}$) $\in$ P $\Leftrightarrow$ $\mathbb{A}$ HAS ALGEBRAIC INVARIANT

$$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$$

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ALGEBRAIC INVARIANTS = POLYMORPHISMS

$\mathbb{A} = (A\,;\,R_1,\dots,R_m)$ STRUCTURE

$f : A^{\ell} \to A$ POLYMORPHISM $:\Longleftrightarrow$

f HOMOMORPHISM $\mathbb{A}^{\ell} \to \mathbb{A}$ $\Longleftrightarrow$

$\forall i : \forall F_1 \dots F_{\ell} \in R_i$

$$f\Big(\overset{\shortparallel}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}} \cdots \overset{\shortparallel}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}\Big) \in R_i$$

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$$f\Big(\overset{''}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}} \cdots \overset{''}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}\Big) \in R_i$$

$\boxed{\mathrm{Pol}(\mathbb{A})} := \{ f \mid f \text{ POLYMORPHISM OF } \mathbb{A} \}$

- CONTAINS PROJECTIONS $(x_1 \dots x_\ell) \mapsto x_j$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS OF AN ALGEBRA ON A !

EXAMPLE

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{ccccc} \min(& x_1 & y_1 &)= & \\ & \wedge & \wedge & & \wedge \\ \min(& x_2 & y_2 &)= & \end{array}$$

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- $(x,y,z) \mapsto x-y+z \quad \in \mathrm{Pol}(\mathbb{Z}_n, \{0\}, \{1\}, \overline{+})$
  - $0-0+0 = 0$
  - $1-1+1 = 1$
  - $$\underset{\in \overline{+}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} - \underset{\in \overline{+}}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} + \underset{\in \overline{+}}{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}} = \underset{\in \overline{+}}{\begin{pmatrix} \\ \\ \end{pmatrix}}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

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  SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

↑ NOT GREAT ☹

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(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69) GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\operatorname{Pol}(\mathbb{A}) \subseteq \operatorname{Pol}(\mathbb{B}) \xrightarrow{①} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\xrightarrow{②}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

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---

② **PROOF**:

LET $\varphi \equiv \exists x_1 \ldots x_n \, C_{i_1}(\;) \wedge \ldots \wedge C_{i_\ell}(\;)$ BE AN INSTANCE OF CSP($\mathbb{B}$)

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION IN $\mathbb{A}$

$\rightsquigarrow \tilde{\varphi}$!

$$\mathbb{A} \models \tilde{\varphi} \iff \mathbb{B} \models \varphi$$

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{l}\min(x_1 \quad y_1) = \\ \qquad \wedge \qquad \wedge \qquad\qquad \wedge \\ \min(x_2 \quad y_2) = \end{array}$$

- $(x,y,z) \mapsto x-y+z \in \mathrm{Pol}(\mathbb{Z}_n, \{0\}, \{1\}, \overline{+})$
  - $0-0+0=0$
  - $1-1+1=1$
  - $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$, each $\in \overline{+}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

↑ NOT GREAT

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69]
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Longrightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Longrightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

---

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① **PROOF**:

LET R BE RELATION

SHOW: R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\Leftrightarrow$

R PP-DEFINABLE IN $\mathbb{A}$

R INVARIANT UNDER Pol(A)

$\Leftrightarrow$

R PP-DEFINABLE IN A

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\Leftrightarrow$

R PP-DEFINABLE IN $\mathbb{A}$

"$\Leftarrow$" EASY: $S(x_1,\dots x_l)$, $T(y_1\dots,y_n)$ INVARIANT

$\Rightarrow$ $S \wedge T$ INVARIANT,

$\exists x_i\, S(x_1\dots x_l)$ INVARIANT

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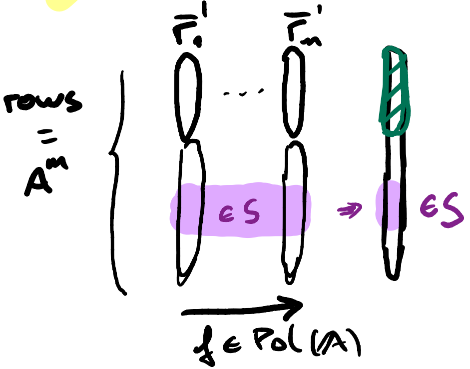
$\exists x_i\, S(x_1 \dots x_l)$ INVARIANT

"$\Rightarrow$" LET $R = \{\bar{t}_1 \dots \bar{t}_m\}$ ($\mathbb{A}$ FINITE!)



$R' := \{f(\bar{t}_1' \dots \bar{t}_m') \mid f \in \mathrm{Pol}(\mathbb{A})\}$

R' IS $\{\wedge, =\}$-DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

$\square$

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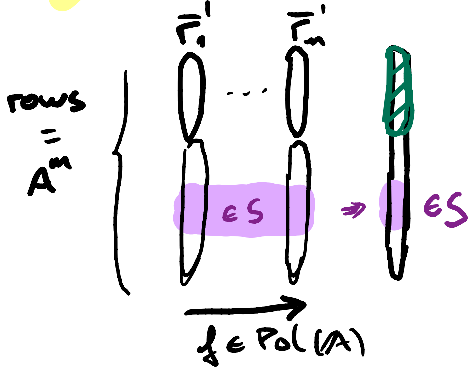

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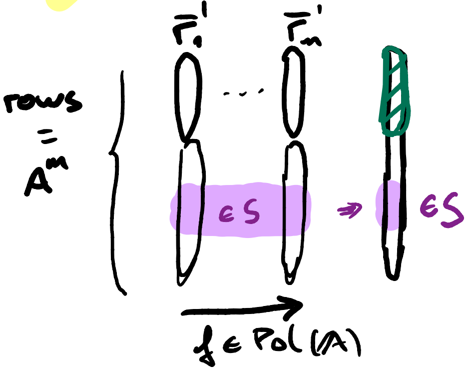

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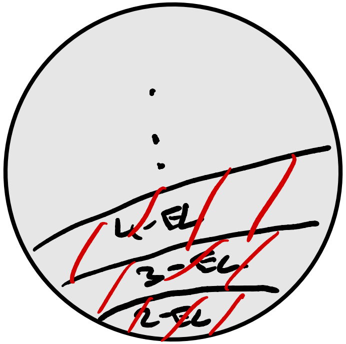
POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



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$\Longleftrightarrow$

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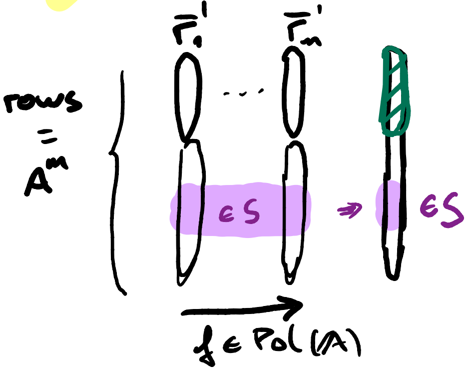

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□

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69, GEIGER '68]

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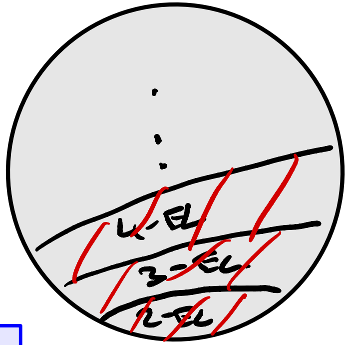

THEOREM

(POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ Pol($\mathbb{A}$) CONTAINS:

- CONSTANT
- max
- min
- maj
- minority

OR Pol($\mathbb{A}$) = {"UNARY" INJECTIONS}

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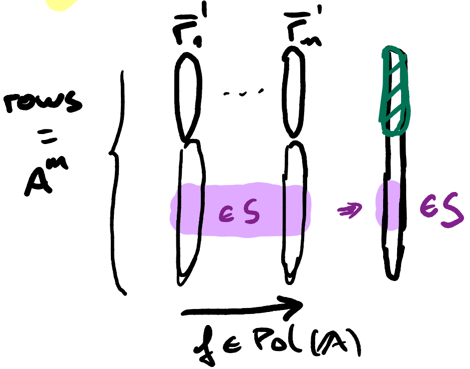

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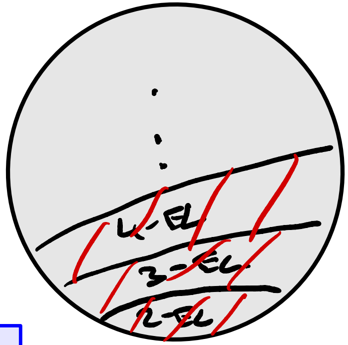

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THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

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  $x-y+z$!

  INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

  GAUSS ALGORITHM

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- min (AND DUALLY max)

  INVARIANT REL. HAVE SMALLEST TUPLE

  SET EACH VARIABLE TO 0 UNLESS SOME CONSTRAINT REQUIRES 1

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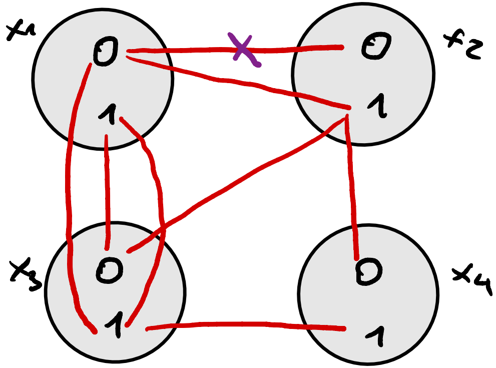
- MAJORITY $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
  LOCAL CONSISTENCY ALGORITHM
  (2,3)- MINIMALITY "

- KEEP LISTS OF POSSIBLE VALUES FOR PAIRS OF VARIABLES (EDGES)
- REMOVE EDGES LOOKING AT CONSISTENCY OF THE LISTS FOR 3 VARIABLES AT A TIME



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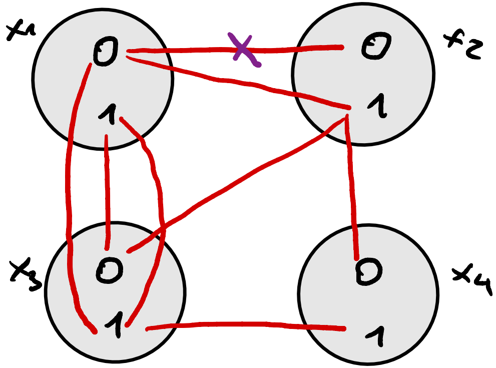


EXERCISE

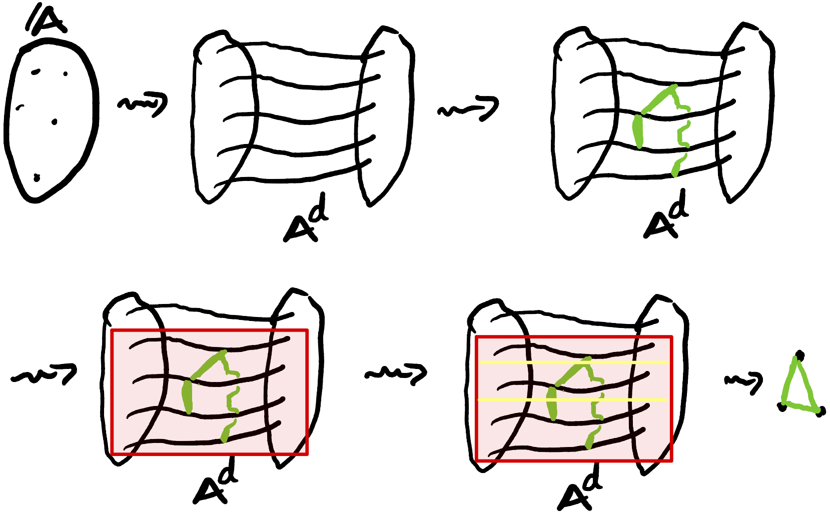
SHOW LOCAL CONSISTENCY SOLVES CSP CORRECTLY IF SOLUTIONS INVARIANT UNDER MAJORITY

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$ $\iff$ $\mathbb{B}$ IS A PP-DEFINABLE FACTOR OF PP-DEFINABLE SUBSTRUCTURE OF PP-DEFINABLE STRUCTURE ON FINITE POWER OF $\mathbb{A}$



PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

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OF PP-DEFINABLE SUBSTRUCTURE
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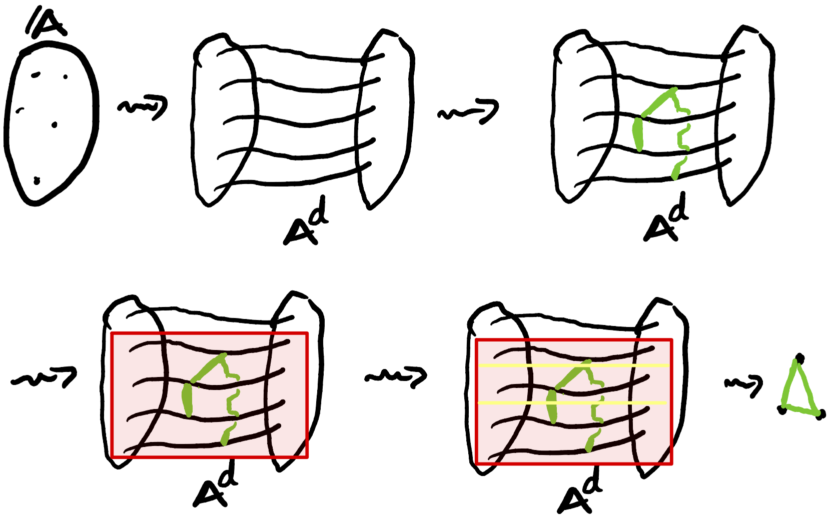

EXAMPLE

$$(\mathbb{Z},+,\cdot,0,1) \rightsquigarrow (\mathbb{Z}^2,\tilde{+},\cdot,(0,1),(1,1)) \rightsquigarrow$$

$$\rightsquigarrow (\mathbb{Z}\times(\mathbb{Z}\setminus\{0\}),\tilde{+},\cdot,(0,1),(1,1))$$

$$\rightsquigarrow (\mathbb{Q},+,\cdot,0,1)$$

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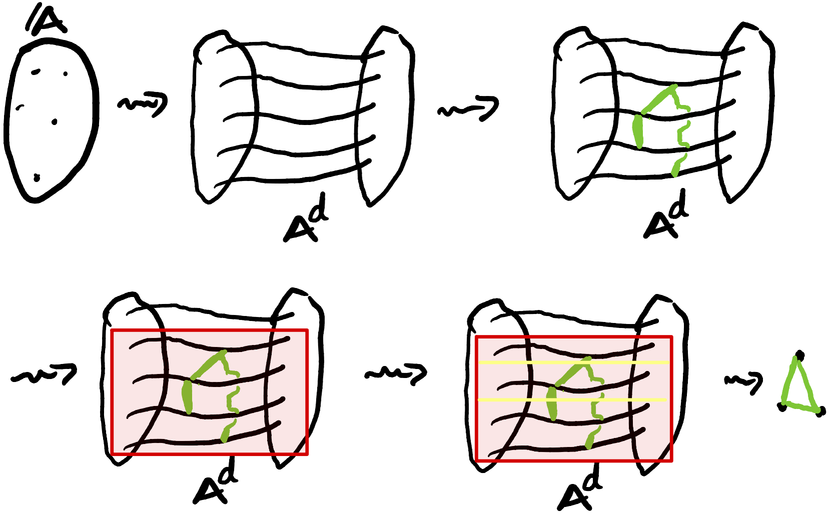

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$$\rightsquigarrow (\mathbb{Q},+,\cdot,0,1)$$

EXERCISE

- SHOW THIS IS A PP-INTERPRETATION
- PP-INTERPRET $(\mathbb{Q},<)$

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$ $:\Leftrightarrow$

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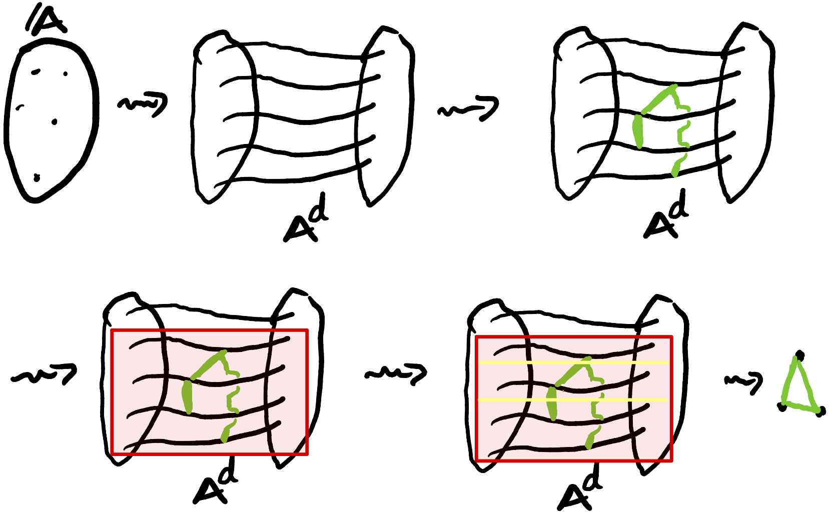

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POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

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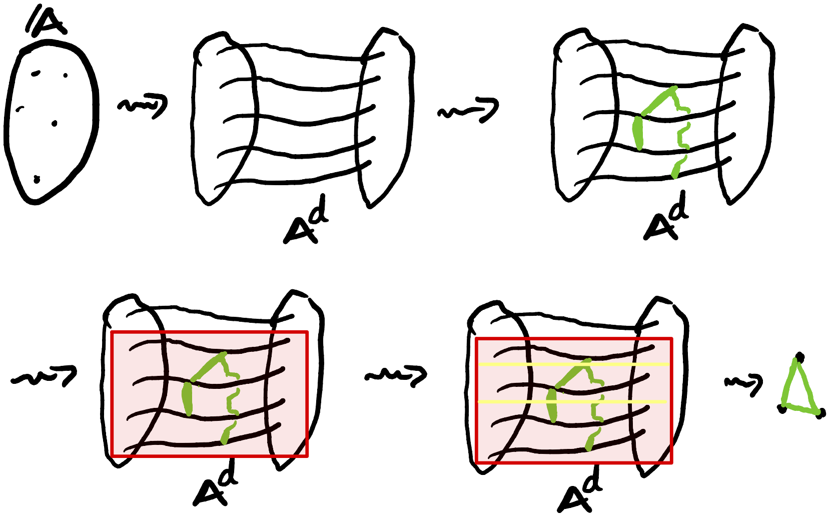

EXAMPLE

$$(\mathbb{Z}, +, \cdot, 0, 1) \rightsquigarrow (\mathbb{Z}^2, \tilde{+}, \cdot, (0,1), (1,1)) \rightsquigarrow$$

$$\rightsquigarrow (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0,1), (1,1))$$

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EXERCISE
- SHOW THIS IS A PP-INTERPRETATION
- PP-INTERPRET $(\mathbb{Q}, <)$

POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

EQUATIONAL CONDITIONS:

- $\exists f \in \mathrm{Pol}(\mathbb{Q}, <)$ $\forall x, y$ $f(x,y) = f(y,x)$
- $\exists f \in \mathrm{Pol}(\mathbb{Z}, \{0\}, \{1\}, +)$
  $\forall x, y, z$ $f(x,x,y) = f(y,x,x) = y$
- $\exists f \in \mathrm{Pol}(K_3)$: $\forall x$ $f f f(x) = x$

DEFINITION

- EQUATIONAL („STRONG MALCEV") CONDITION:

SENTENCE

$$\Phi = \exists f_1 \exists f_2 \cdots \forall x_1 \forall x_2 \cdots\cdot$$

$$\begin{aligned} & s_1(\text{variables}) = t_1(\text{variables}) \\ \wedge & \\ & \vdots \\ \wedge\ & s_k(\text{variables}) = t_k(\text{variables}) \end{aligned}$$

WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \ldots$

DEFINITION

- EQUATIONAL („STRONG MALCEV") CONDITION: SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots\dots$$

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- Ψ TRIVIAL $:\Leftrightarrow$ Pol(ANY STRUCTURE) $\models \Psi$

  $\Leftrightarrow$ Ψ SATISFIABLE BY PROJECTIONS

DEFINITION

- EQUATIONAL („STRONG MALCEV") CONDITION: SENTENCE

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  $\Leftrightarrow$ Ψ SATISFIABLE BY PROJECTIONS

- $\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) := \{\Psi \mid \mathrm{Pol}(\mathbb{A}) \models \Psi\}$

DEFINITION

- EQUATIONAL ("STRONG MALCEV") CONDITION: SENTENCE

$$\Psi = \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots \quad s_1(\text{variables}) = t_1(\text{variables}) \wedge \vdots \wedge s_k(\text{variables}) = t_k(\text{variables})$$

WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL :$\Leftrightarrow$ Pol(ANY STRUCTURE) $\models \Psi$

  $\Leftrightarrow$ Ψ SATISFIABLE BY PROJECTIONS

- $EQ(Pol(\mathbb{A})) := \{\Psi \mid Pol(\mathbb{A}) \models \Psi\}$

(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KROKHIN '00)

$EQ(Pol(\mathbb{A})) \subseteq EQ(Pol(\mathbb{B}))$

$\overset{①}{\Rightarrow}$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

DEFINITION

- **EQUATIONAL ("STRONG MALCEV") CONDITION:**

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---

② **PROOF**: AS WITH PP-DEFINITIONS

GIVEN INSTANCE OF CSP($\mathbb{B}$):

$$\varphi = \exists x_1 \ldots x_n \; C_{i_1}(\;) \wedge \ldots \wedge C_{i_m}(\;)$$

- FOR EACH OCCURRENCE OF x_i CREATE NEW VARIABLES $x_i^1 \ldots x_i^d$ (d ... POWER)
- ADD CONSTRAINTS THAT ALL TUPLES $(x_i^1 \ldots x_i^d)$ BE EQUIVALENT
- ADD CONSTRAINTS RESTRICTING $(x_i^1 \ldots x_i^d)$ TO SUBSET
- TRANSLATE CONSTRAINTS ON x_i INTO CONSTRAINTS ON x_i^d

EXERCISE CHECK THIS WORKS

① PROOF SKETCH OF:

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DEFINITION

$\mathbb{A}, \mathbb{B}$ RELATIONAL STRUCTURES

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EXERCISE: SHOW THE CONVERSE

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-~~DEFINABILITY~~ INTERPRETABILITY

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEF~~INAB~~ILITY INTERPRETABILITY

EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$

CONSIDER

$$R = \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$$

$CSP(\{0,1\}, R)$ IS 1-IN-3SAT

(NP-COMPLETE)

$$\begin{tikzcd} Pol(\{0,1\}, R) = \{f : f \text{ PROJECTION}\} \arrow[d, "\text{HOM}"] \\ Pol(\mathbb{A}) \end{tikzcd}$$

EXERCISE SHOW IT.

$Pol(\mathbb{A})$ FOR ANY $\mathbb{A}$!

STRONGEST WRT PP-INTERPRETATIONS!

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEF~~INA~~BILITY

INTERPRETABILITY

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EXAMPLE

$CSP(K_3)$ IS 3-COLORING

CLAIM $\forall f(x_1 \dots x_n) \in Pol(K_3)$

$\exists i \; \exists g \in Aut(K_3)$

$f(x_1, \dots x_n) = g(x_i)$

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FOR ANY $\mathbb{A}$!

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COROLLARY

$$\varphi : \mathrm{Pol}(K_3) \to \mathrm{Pol}(\{0,1\}, R)$$

$$f \mapsto i\text{-th PROJ}$$

CLONE HOMOMORPHISM

$\Rightarrow$ K_3 PP-INTERPRETS $(\{0,1\},R)$ AND VICE-VERSA

$\Rightarrow$ 3-COLORING IS NP-COMPLETE

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PROOF

CONSIDER $h(x) := f(x, \dots, x) \in Aut(K_3)$

f, $h \circ f$, OR $h^2 \circ f$ IS IDEMPOTENT

$\Rightarrow$ CALL THIS FUNCTION g

SHOW $\exists i \ g(x_1 \dots x_n) = x_i$

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$n=1$ ✓

$n=2$

$\{1\}$ INVARIANT

$\Rightarrow \{2,3\}$ INVARIANT

SIMILARLY $\{1,3\}, \{1,2\}$

$g(1,2) - g(2,1)$, $g(1,1) = 1$, $g(2,2) = 2$

$\Rightarrow g|_{\{1,2\}}$ PROJECTION!

$$\left.\begin{array}{l} \ g(1,2) = 1 \\ \ \ \ |\ \ \ | \qquad | \\ \Rightarrow g(3,1) = 3 \end{array}\right\} \Rightarrow \text{PROJECTIONS COINCIDE!}$$

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n>2 $A \subseteq \{1, \dots, n\}$ STRONG :⇔

$$g(y \cdots y \underbrace{x \cdots x}_{\text{INDICES IN } A} y \cdots y) = x$$

- $\{1, \dots, n\}$ STRONG
- $\forall A$ (A STRONG OR $\{1, \dots, n\} \setminus A$ STRONG)
- $\forall A, B$ A, B STRONG ⇒ $A \cap B$ STRONG

A B

$g(0, \dots, 0, 1, \dots, 1) = 0$

$g(1 \cdots 1\ 2 \cdots\cdots 2) = 2$

⇒ $g(2 \ \ 2\ 1 \cdots 1\ 0 \cdots 0) = 1$

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⇒ $g(2 \ 2 \ 1 \cdots 1 \ 0 \cdots 0) = 1$

⇒ $g(1 \cdots 1 \ 0 \cdots 0 \ 1 \cdots 1) = 0$

STRONG SUBSETS ULTRAFILTER

⇒ $\exists i$ $\{i\}$ STRONG WLOG $i=1$

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PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

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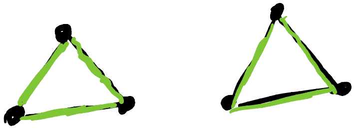
EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$



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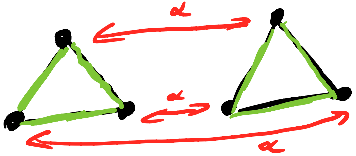

LET $S(x,y,z) = \begin{cases} z, & x,y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \mathrm{Pol}(\mathbb{A})$

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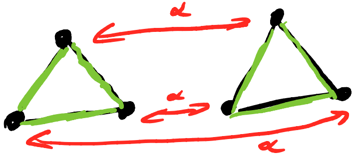

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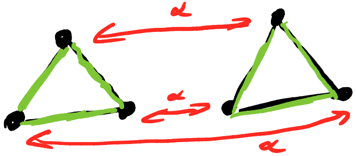

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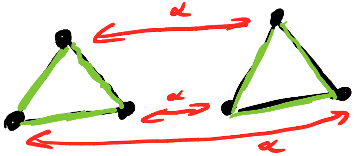

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PROOF

"$\to$"

$$\begin{tikzcd} & \mathbb{X} \arrow[dl] \arrow[dr] & \\ \mathbb{A} \arrow[rr, leftrightarrow] & & \mathbb{B} \end{tikzcd}$$

"$\leftarrow$" $\mathbb{A} \in \mathrm{CSP}(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

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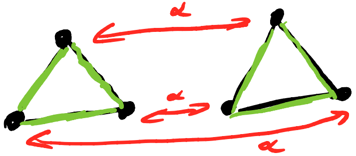

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"⇒"

$$\begin{tikzcd} & \mathbb{X} \arrow[dl] \arrow[dr] & \\ \mathbb{A} \arrow[rr, leftrightarrow] & & \mathbb{B} \end{tikzcd}$$

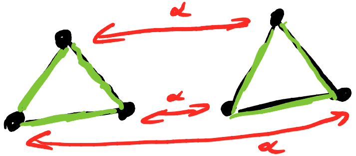
"⇐" $\mathbb{A} \in CSP(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$



LET $s(x,y,z) = \begin{cases} z, & x,y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \mathrm{Pol}(\mathbb{A})$
AND $s(x,x,y) = y = s(y, \alpha y, x)$
$\Rightarrow \mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(K_3)$

BUT $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(K_3)$!

SINCE $\mathbb{A} \overset{\text{HOM}}{\Rightarrow} K_3$, $K_3 \overset{\text{HOM}}{\Rightarrow} \mathbb{A}$

DEFINITION

$\mathbb{A}, \mathbb{B}$ RELATIONAL STRUCTURES
HOMOMORPHICALLY EQUIVALENT $:\Leftrightarrow$
$\exists h: \mathbb{A} \to \mathbb{B}$ HOMOMORPHISM & VICE-VERSA

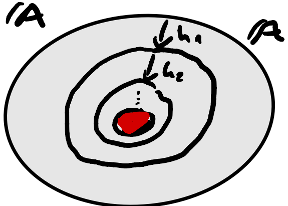
$\mathbb{A}, \mathbb{B}$ H.E. $\Leftrightarrow$ $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{B})$

PROOF

"$\Rightarrow$"
$$\begin{tikzcd} & \mathbb{X} \arrow[dl] \arrow[dr] & \\ \mathbb{A} \arrow[rr, leftrightarrow] & & \mathbb{B} \end{tikzcd}$$

"$\Leftarrow$" $\mathbb{A} \in \mathrm{CSP}(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD
- EACH EQUIVALENCE CLASS HAS UNIQUE SMALLEST REPRESENTATIVE: THE CORE OF $\mathbb{A}$



DEFINITION & THEOREM

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\iff$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP.INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \cdots \rightsquigarrow \mathbb{B}$$

DEFINITION & THEOREM

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\Longleftrightarrow$

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$$\Longleftrightarrow$$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C} \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$$

DEFINITION & THEOREM

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$$\Longleftrightarrow$$

$$\mathbb{A} \overset{\text{PP-DEF.}}{\rightsquigarrow} \mathbb{C} = (A^d, R_1 \dots R_n) \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$$

DEFINITION & THEOREM

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\Longleftrightarrow$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

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$$\Longleftrightarrow$$

$\exists \varphi : Pol(\mathbb{A}) \to Pol(\mathbb{B})$ MINION HOM:

- PRESERVES ARITIES
- PRESERVES MINORS

$$\varphi(f(x,\dots,x,y\dots y)) =$$
$$= \varphi(f)(x\dots x\ y\dots y)$$

DEFINITION & THEOREM

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\Longleftrightarrow$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \cdots \rightsquigarrow \mathbb{B}$$

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ALL HEIGHT 1 EQUATIONS SATISFIED IN $\mathrm{Pol}(\mathbb{A})$ ARE SATISFIED IN $\mathrm{Pol}(\mathbb{B})$

DEFINITION & THEOREM

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$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

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ALL HEIGHT 1 EQUATIONS SATISFIED IN $\mathrm{Pol}(\mathbb{A})$ ARE SATISFIED IN $\mathrm{Pol}(\mathbb{B})$

EXAMPLE

HEIGHT 1:

- $m(x,x,y) = m(x,y,x) = m(y,x,x)$
- $c(x_1 \dots x_n) = c(x_2 \dots x_n, x_1)$

NOT HEIGHT 1:

- $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
- $s(x,x,y) = y = s(y, xy, x)$

DEFINITION & THEOREM

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\iff$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

$$\iff$$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C} \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$$

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- $s(x,x,y) = y = s(y,xy,x)$

EXAMPLE

$\mathbb{A} = K_3 \dot{\cup} K_3$ PP-CONSTRUCTS K_3

PP-~~INTERPRETS~~

$f \in \text{Pol}(\mathbb{A}) \Rightarrow f|_{K_3}$ IS ESSENTIALLY UNARY

$\Rightarrow \exists$ MINION HOMOMORPHISM $\text{Pol}(\mathbb{A}) \to \text{Pol}(K_3)$

(ADULT) THEOREM

(BARTO+OPRŠAL+P. 16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = f_j(\text{variables})$$

$$\cancel{s_i} \qquad\qquad \cancel{t_j}$$

$$EQ^1(Pol(\mathbb{A})) \leq EQ^1(Pol(\mathbb{B}))$$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

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COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Longrightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Longrightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

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THEOREM $\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

(ADULT) THEOREM

(BARTO + OPRŠAL + P. 16)

$EQ^\wedge$ ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

$\cancel{s_i}$ $\cancel{t_j}$

$EQ^\wedge(Pol(\mathbb{A})) \leq EQ^\wedge(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

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$EQ^\wedge(Pol(\mathbb{A}))$ TRIVIAL

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THEOREM

$EQ^\wedge(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

$$Pol(\mathbb{A}) \models \exists f \; \forall x,y,z \quad f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$$

6-ARY SIGGERS

(SIGGERS '10)

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

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6-ARY SIGGERS (SIGGERS '10)

$\models \exists f\ \forall a,r,e\quad f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS (KEARNES + MARKOVIĆ + MCKENZIE '14)

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4-ARY SIGGERS

(KEARNES + MARKOVIĆ + MCKENZIE '14)

$\models \exists f \; \forall x_1 \cdots x_n$

$f(x_1 \cdots x_n) = f(x_2 \cdots x_n, x_1)$

$\forall n \geq |A|$ PRIME

CYCLIC

(BARTO + KOZIK '11)

(ADULT) THEOREM

(BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

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$\models \exists f \, \forall x_1 \cdots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall \, n > |A|$ PRIME

CYCLIC

(BARTO + KOZIK '11)

$\models \exists f \, \forall x,y \quad f(x \dots xy) = \dots = f(yx \dots x)$

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WNU

(MARÓTI + MCKENZIE '08)

(ADULT) THEOREM

(BARTO+ OPRŠAL+P. '16)

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THEOREM

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6-ARY SIGGERS (SIGGERS '10)

$\models \exists f\ \forall a,r,e\ \ f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS (KEARNES+ MARKOVIĆ +MCKENZIE '14)

$\models \exists f\ \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO +KOZIK '11)

$\models \exists f\ \forall x,y\ \ f(x \dots xy) = \dots = f(yx \dots x)$

$\forall n \geq |A|$ PRIME

WNU (MARÓTI +MCKENZIE '08)

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P (BULATOV, ZHUK '17)

THEOREM $EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL

$\implies CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

THEOREM $EQ^1(Pol(A))$ NON-TRIVIAL

$\Rightarrow CSP(A) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE BY SOLVING LINEAR EQUATIONS OR LOCAL CONSISTENCY CHECKING

THEOREM $\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE BY SOLVING LINEAR EQUATIONS OR LOCAL CONSISTENCY CHECKING

THEOREM (BARTO + BODOR + KOZIK + MOTTET + P. '23)

$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL,
$\mathbb{A}$ CORE

$$\Rightarrow \mathrm{Pol}(\mathbb{A}) \models \exists f \, \exists \alpha_1 \dots \alpha_m \; \forall x,y,z \quad f(x,y,x,z,y,z,x,\dots x) = f(y,x,z,x,z,y,\alpha_1 y,\dots \alpha_m y)$$

THEOREM $EQ^1(Pol(A))$ NON-TRIVIAL

$\implies CSP(A) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
OR LOCAL CONSISTENCY CHECKING

THEOREM (BARTO + BODOR + KOZIK + MOTTET + P. '23)

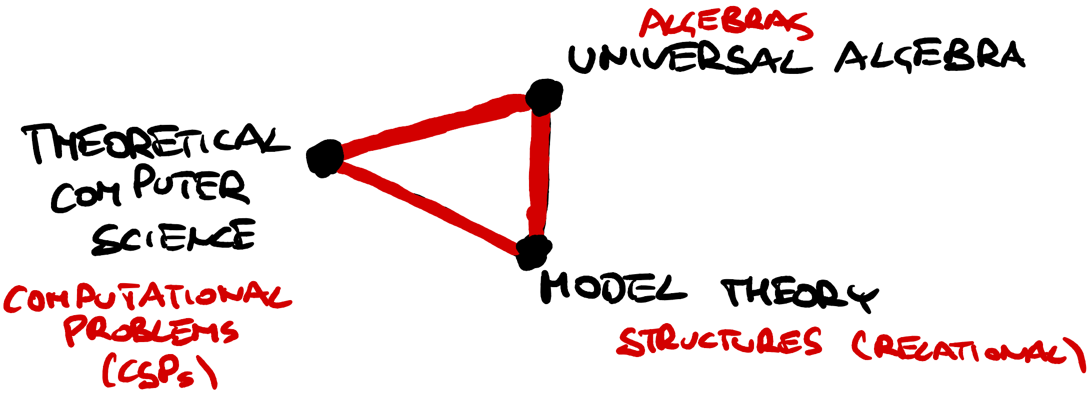
$EQ(Pol(A))$ NON-TRIVIAL,
A CORE

$$\implies Pol(A) \models \exists f\ \exists \alpha_1 \dots \alpha_m\ \forall x,y,z\quad f(x,y,x,z,y,z,x,\dots x) = f(y,x,z,x,z,y,\alpha_1 y,\dots \alpha_m y)$$

OPEN PROBLEM

$EQ(Pol(A))$ NON-TRIVIAL
A NOT A CORE

$\implies$?



PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs

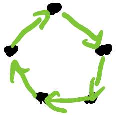
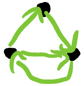
- ALGEBRAS
- RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.

- ALGEBRAS
- RELATIONAL STRUCTURES

EXERCISES

- CSP () ALGORITHM? NON-TRIVIAL H1 EQUATION?
- CSP () —— " ——
- CSP $\left(\{0,1\}; \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}\right)$
- CSP ()

PART II : MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

FINITE-DOMAIN CSPs : RATHER LIMITED

FINITE-DOMAIN CSPs : RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD

$\in$ NP

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in NP$

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

- n-COLORING PROBLEM

GENERALIZATION: H-COLORING PROBLEM

H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '91)

FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP

- SOLVING **(LINEAR) EQUATIONS** / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: **RESTRICTIONS OF SAT**

  $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

  SATISFYING ASSIGNMENT?
- **n-COLORING PROBLEM**

  GENERALIZATION: **H-COLORING PROBLEM**

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL+ NEŠETŘIL '91)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

FINITE-DOMAIN CSPs: RATHER LIMITED

∈ NP

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

  $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

  SATISFYING ASSIGNMENT?

- n-COLORING PROBLEM

  GENERALIZATION: H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '91)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

  $$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

  SATISFYING ASSIGNMENT?

- n-COLORING PROBLEM

  GENERALIZATION: H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '90)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}$ / $\mathbb{Z}$
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FINITE-DOMAIN CSPs: RATHER LIMITED

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EXERCISE

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  $\mathcal{F}$ ... FINITE SET OF TOURNAMENTS

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MMSNP : τ ... RELATIONAL SIGNATURE , σ ... UNARY PREDICATES

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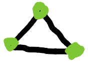
EXAMPLE $\mathcal{F} = \{$  ,  $\}$ NO-MONOCHROMATIC TRIANGLE

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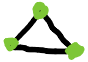
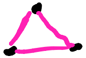
EXAMPLE $\mathcal{F} = \{$  ,  $\}$ NO-MONOCHROMATIC TRIANGLE

GMSNP : $\mathcal{F} = \{$  ,  $\}$ —— " —— (COLORING EDGES)

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WHY ALL THOSE FORBIDDEN PATTERN PROBLEMS?

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NP = ESO : EXISTENTIAL SECOND ORDER LOGIC (FAGIN '73)

τ ... SIGNATURE, K ... ISOMORPHISM-CLOSED CLASS OF FINITE τ-STRUCTURES

$\Rightarrow$ $K \in NP \Leftrightarrow K$ DEFINABLE BY ESO-SENTENCE :

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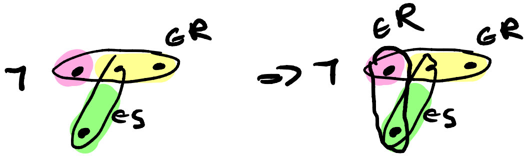
PATTERNS CONNECTED, MONOTONE ⇒ K IS A CSP

= CLOSED UNDER HOMOMORPHISMS: τ-ATOMS ONLY APPEAR POSITIVELY



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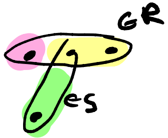

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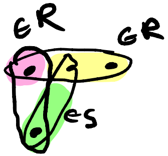

EXERCISE

WRITE DOWN NO MONO-CHROMATIC TRIANGLE

WHEN DO POLYMORPHISMS WORK?

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DEFINITION

$\mathbb{A}$ COUNTABLE

$\mathbb{A}$ ω- CATEGORICAL $:\Leftrightarrow$

$\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ FINITELY MANY ORBITS
FOR ALL $n \geq 1$

"$\mathbb{A}$ INFINITE BUT $\mathbb{A}^n / \mathrm{Aut}\,\mathbb{A}$ FINITE"

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EXAMPLES

- $(\mathbb{N}, =)$
- $(\mathbb{Q}, <)$
- RANDOM GRAPH (V, E)
- COUNTABLE VECTOR SPACE OVER FINITE FIELD

× $(\mathbb{Z}, <)$, $(\mathbb{N}, <)$

× COUNTABLE VECTOR SPACE OVER $\mathbb{Q}$

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SPOILER

$\mathbb{A}$ ω- CATEGORICAL $\Rightarrow$

- PP-DEFINABILITY DEPENDS ONLY ON $\mathrm{Pol}(\mathbb{A}$
- PP- INTERPRETABILITY OF FINITE STRUCTURES DEPENDS ON LOCAL EQUATIONS OF $\mathrm{Pol}(\mathbb{A}$
- PP- CONSTRUCTABILITY OF FINITE STRUCTURES : LOCAL h1-EQUATIONS OF $\mathrm{Pol}(\mathbb{A})$
- $\mathbb{A}$ HAS A "CORE"
- $\exists \mathbb{A}$ ω-CATEGORICAL, CSP($\mathbb{A}$) UNDECIDABLE, $\mathbb{A}$ DOES NOT PP-CONSTRUCT K_3
- $\exists$ NICE SUBCLASS OF ω-CAT. WITH P/NP-COMPLETE CONJECTURE ("FORBIDDEN PATTERN PROBLEMS FOR FINITE SETS OF PATTERNS")

CONSTRUCTING ω-CATEGORICAL STRUCTURES

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HP: $\forall \mathbb{F} \leq \mathbb{G} \in \mathcal{K} : \mathbb{F} \in \mathcal{K}$

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$\exists \mathbb{G} \in \mathcal{K} \; \exists e_i : \mathbb{F}_i \to \mathbb{G}$ EMBEDDINGS

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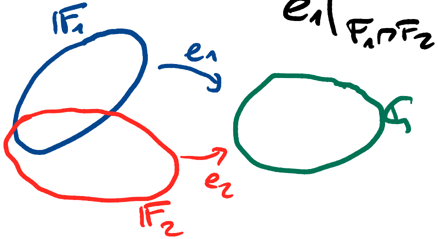

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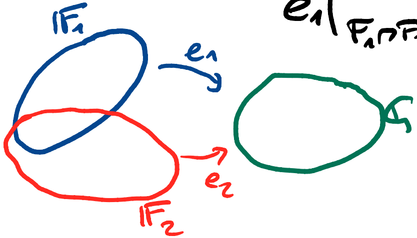

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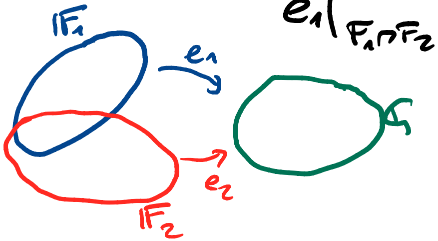

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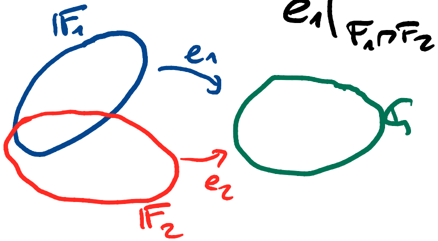

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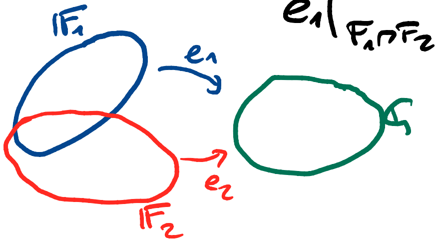

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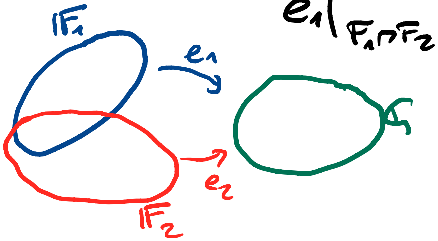

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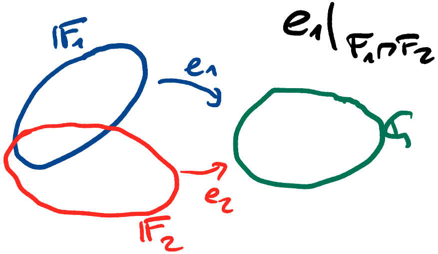

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- GRAPHS, DIGRAPHS, TOURNAMENTS, POSETS, ...

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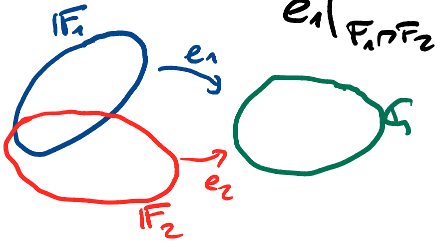

$\Rightarrow \exists! \mathbb{A}$:

- $\mathbb{K} = \mathrm{Age}(\mathbb{A}) = \{\mathbb{G} \leq \mathbb{A} : \mathbb{G} \text{ finite}\}$
- $\mathbb{A}$ HOMOGENEOUS: EVERY FINITE PARTIAL ISO ON $\mathbb{A}$ EXTENDS TO AN AUTOMORPHISM

- $\mathbb{A}$ "GENERIC" REPRESENTATION OF $\mathbb{K}$ IN ONE STRUCTURE

  "FRAÏSSÉ-LIMIT OF $\mathbb{K}$"
- $\mathbb{A}$ ω-CATEGORICAL $\Leftrightarrow$ $\forall n \ \{\mathbb{A} \in \mathbb{K} \mid |A| = n\}/_{iso}$ FINITE
- E.G. IF τ FINITE
- EVERY ω-CATEGORICAL STRUCTURE IS FIRST-ORDER DEFINABLE IN A FRAÏSSÉ LIMIT

  "FIRST-ORDER REDUCT OF $\mathbb{A}$"

EXAMPLE

- $(\mathbb{Q}, <)$ = LIMIT OF FINITE LINEAR ORDERS

  👁 $\mathrm{CSP}(\mathbb{Q}, <) = \mathrm{CSP}(\mathbb{N}, <)$
- GRAPHS, DIGRAPHS, TOURNAMENTS, POSETS, …

EXERCISE: NON-TRIVIAL CLASS OF GRAPHS W/ HP, AP

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

- $\mathrm{CSP}(\mathbb{Q},<)$

  GIVEN FINITE DIGRAPH $\mathbb{D}$, $\mathbb{D} \overset{?}{\to} (\mathbb{Q},<)$

  $\mathbb{D} \overset{?}{\to}$ FINITE LINEAR ORDER ?

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

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- $\mathrm{CSP}(\mathbb{Q}, B(x,y,z))$

  $B(x,y,z) :\Leftrightarrow (x<y \wedge y<z) \vee$
  $(z<y \wedge y<x)$

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

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  EXERCISE: COMPLEXITY ?

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

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CSPs OF FO-REDUCTS OF $(\mathbb{Q},<)$

"LINEAR-ORDER-SAT", "TEMPORAL CSPs"

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"LINEAR-ORDER-SAT", "TEMPORAL CSPs"

THEOREM

- LINEAR-ORDER-SAT $(\mathbb{A})$ IN P UNLESS $\mathbb{A}$ PP-CONSTRUCTS K_3 (BODIRSKY + KÁRA '07)

- GRAPH-SAT (BODIRSKY + P. '11)
- POSET-SAT (VAN PHAM + KOMPATSCHER '16)
- TOURNAMENT-SAT (MOTTET + P. '21)
- HYPERGRAPH-SAT (MOTTET + NAGY + P. 23)

  ⋮

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

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CONJECTURE (BODIRSKY + P. '11)

TRUE FOR ALL FIRST-ORDER REDUCTS OF HOMOGENEOUS STRUCTURES GIVEN BY FINITELY MANY FORBIDDEN SUBSTRUCTURES

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EXAMPLE

- LINEAR ORDERS = Forb(, , , , ...)
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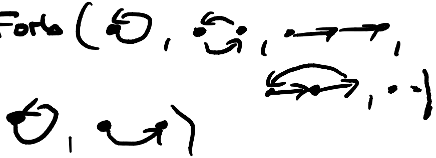

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

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LET $\mathbb{H}$ FINITE GRAPH $\Rightarrow$

$$\mathbb{H} \in CSP(\mathbb{G}) \Leftrightarrow \mathbb{H} \xrightarrow{\text{HOM}} \mathbb{G}$$

$$\overset{\text{Ex.}}{\Leftrightarrow} \mathbb{H} \xrightarrow{\text{EMB}} \mathbb{G}$$

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(FELLER + P. '24)

$CSP(\mathbb{G}) \in P$ UNLESS $\mathbb{G}$ PP-CONSTRUCTS K_3

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WHAT IF THE CLASS OF SOLUTIONS TO A PROBLEM DOES NOT HAVE AP?

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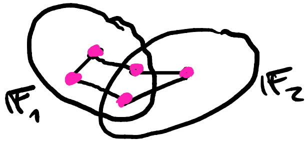
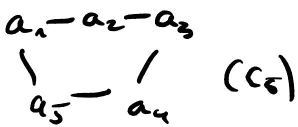
EXAMPLE

GRAPHS WITHOUT $a_1 \cdots a_5$:





OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AP, THEN TAKE REDUCT

THE GRAPH ORIENTATION PROBLEM AS CSP

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Ex. $\Leftrightarrow \mathbb{H} \xrightarrow{emb} \mathbb{G}$

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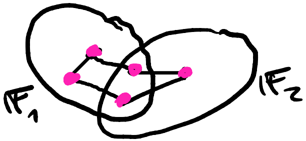
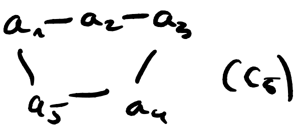




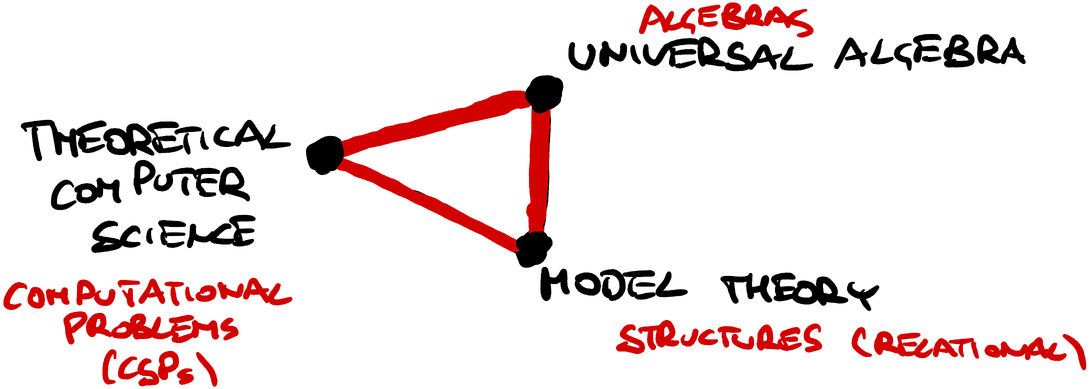
OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AP, THEN TAKE REDUCT

THEOREM [CHERLIN-SHELAH-SHI]

$\mathcal{F}$... FINITE SET OF STRUCTURES, CONNECTED, HOM-CLOSED

$\Rightarrow \exists \mathbb{A}$ ω-CATEGORICAL, FO-REDUCT OF HOMOGENEOUS STRUCTURE GIVEN BY FINITE SET OF FORBIDDEN STRUCTURES

$CSP(\mathbb{A}) = Forb(\mathcal{F})$



PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs.

ALGEBRAS

RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.

ALGEBRAS

RELATIONAL STRUCTURES

EXERCISES

- LET $G = (V,E)$ GRAPH $\quad P(x,y) :\Leftrightarrow \exists a\ E(x,a) \wedge E(a,y)$

  $Q(x,y) :\Leftrightarrow \exists a,b\ E(x,a) \wedge E(a,b) \wedge E(b,y)$

  $\mathcal{K} := \{ G \mid G \text{ FINITE},\ C_5 \overset{\text{HOM}}{\not\to} G \}$

  SHOW: THE STRUCTURES OF $\mathcal{K}$, EXPANDED BY P AND Q, HAVE THE AP

- $\mathcal{F}$ ... (POSSIBLY INFINITE) SET OF FINITE TOURNAMENTS

  SHOW: $\mathcal{K} := \{ D \mid D \text{ DIGRAPH},\ \mathcal{F}\text{-FREE} \}$ HAS AP

  CONCLUDE: THERE EXIST UNDECIDABLE ω-CATEGORICAL CSPs.

- WHEN IS A CLASS OF STRUCTURES A CSP?

PART III : THE MATHEMATICS OF ω-CATEGORICAL CSPs

SHORT VERSION:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

SHORT VERSION:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

(BABY) THEOREM (BODIRSKY + NEŠETŘIL '03)

$\mathbb{A}$ ω-CATEGORICAL, $R \subseteq A^m$ RELATION

$\Rightarrow$ R PP-DEFINABLE IN $\mathbb{A}$ $\Leftrightarrow$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

SHORT VERSION:

BASIC THEORY ... AS FOR FINITE STRUCTURES

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PROOF

"$\Rightarrow$" TRIVIAL

"$\Leftarrow$"

R IS A UNION OF ORBITS OF

$\mathrm{Aut}\,\mathbb{A} \curvearrowright A^m$

LET $\bar{r}_1, \dots, \bar{r}_\ell \in R$ BE REPRESENTATIVES OF THESE ORBITS

LET $(\bar{a}_0, \bar{a}_1, \dots)$ ENUMERATION OF A^ℓ

SHORT VERSION:

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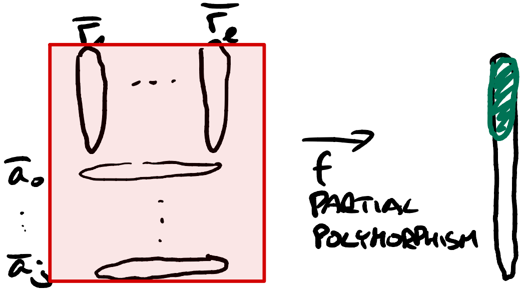

SHORT VERSION:

BASIC THEORY ... AS FOR FINITE STRUCTURES

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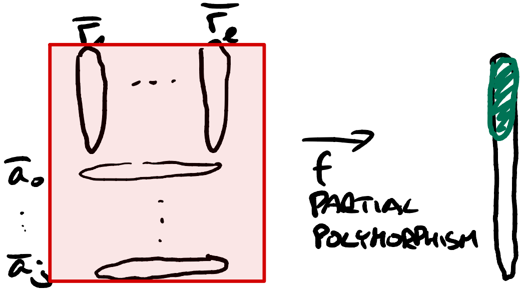

- $R_j \supseteq R$ (PROJECTIONS)

SHORT VERSION:

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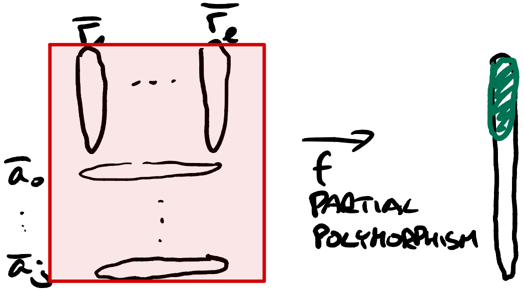

- $R_j \supseteq R$ (PROJECTIONS)
- $R_j \supseteq R_{j+1}$, EVENTUALLY CONSTANT

  (R_j UNION OF ORBITS)

$\Rightarrow$ $\bigcap_j R_j =: \begin{array}{c} R_\infty \\ \cup\!| \\ R \end{array}$ PP-DEFINABLE

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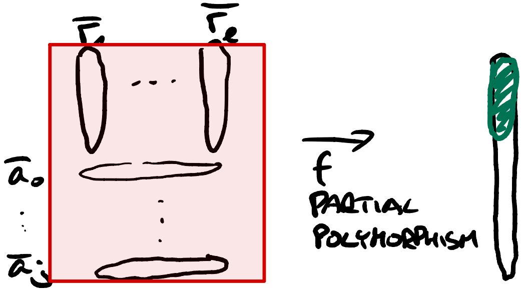

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$\Rightarrow$ $\bigcap_j R_j =: \begin{array}{c} R_\infty \\ \cup \\ R \end{array}$ PP-DEFINABLE

- $\bar r \in R_\infty$

$\Rightarrow \exists f_j$ PART. POLYMORPHISMS WITNESSES

.. $\exists$ SUBSEQUENCE SUCH THAT f_j EXTENDS f_i, $i<j$ MOD. ORBITS

.. COMPOSING WITH AUTOMORPHISMS, f_j EXTENDS f_i $\quad f := \bigcup f_i$

$\Rightarrow \bar r = f(\bar r_1, \dots \bar r_j) \in R$

ADOLESCENT / ADULT THEOREMS:

- CHARACTERISE
  - PP-INTERPRETATIONS BY EQUATIONS / CLONE HOMOMORPHISMS
  - PP-CONSTRUCTIONS BY H1 EQUATIONS / MINION HOMOMORPHISMS

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- ONLY PP-INTERPRET
  CONSTRUCT FINITE $\mathbb{B}$
  IN ω-CATEGORICAL $\mathbb{A}$

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- ONLY PP-INTERPRET / CONSTRUCT FINITE $\mathbb{B}$ IN ω-CATEGORICAL $\mathbb{A}$
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TOPOLOGY

$A^{A^n} = \{f : A^n \to A\}$ CARRIES

TOPOLOGY OF POINTWISE CONVERGENCE:

$$(f_j)_{j\in\omega} \to f \quad \Leftrightarrow$$

$\forall F \subseteq A^n$ FINITE

$f_j|_F = f|_F$ EVENTUALLY

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$Pol(\mathbb{A})$ CLOSED SUBSPACE:

$f \notin Pol(\mathbb{A}) \Rightarrow \exists F \subseteq A$ FINITE:

$$O_{f|_F} = \{g \mid g|_F = f|_F\} \cap Pol(\mathbb{A}) = \emptyset$$

THEOREM (ADOLESCENT + ADULT)

BODIRSKY + P. 45

BARTO + OPRŠAL + P. 17

$\mathbb{A}$ ω-CATEGORICAL, $\mathbb{B}$ FINITE $\Rightarrow$

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① $\mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{CLONE HOMOMORPHISM}]{\text{u.c}} \mathrm{Pol}(\mathbb{B}) \iff \mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

$\iff$ $\exists F \subseteq A$ FINITE :

$$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})|_F) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$$

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„PROOF“ OF ①

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⇐ EASY

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„PROOF" OF ①

⟸ EASY

⟹ $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

$$\xRightarrow{\text{u.c}} \mathrm{Pol}(\mathbb{B}) \supseteq \mathrm{Pol}(\mathbb{C}) \in \mathrm{HSP}^{\text{FIN}}(\mathrm{Pol}(\mathbb{A}))$$

⟹ ... AS IN FINITE CASE

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⟹ ... AS IN FINITE CASE

WHAT IF $\mathrm{Pol}(\mathbb{A}) \cancel{\xrightarrow[\text{MINION}]{\text{u.c}}} \mathrm{Pol}(K_3)$?

THEOREM (ADOLESCENT + ADULT)

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$\Leftarrow$ EASY

$\rightarrow$ $\mathrm{Pol}(\mathbb{A}) \rightarrow \mathrm{Pol}(\mathbb{B})$

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$\Rightarrow$ ... AS IN FINITE CASE

WHAT IF $\mathrm{Pol}(\mathbb{A}) \cancel{\xrightarrow[\text{MINION}]{\text{u.c}}} \mathrm{Pol}(\mathbb{K}_3)$?

RECALL $\mathbb{A}$ FINITE $\Rightarrow$

Pol($\mathbb{A}$) HAS

- CYCLIC OPERATION $c(x_1 \cdots x_n) = c(x_2 \cdots x_n x_1)$
- WNU OPERATION $w(x \cdots x y) = \dots = w(y x \cdots x)$
- 4-ARY SIGGERS OPERATION $s(a,r,e,a) = s(r,a,r,e)$
- 6-ARY SIGGERS OPERATION $s(xyxzyz) = s(yxzxzy)$

...

FALSE FOR ω-CATEGORICAL CA !

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega \underset{\cdot}{=} (\mathbb{N}, \neq)$

ω-CATEGORICAL

$\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

CSP(A) $\in$ P

- DOES $\mathrm{Pol}(\mathbb{A})$ HAVE CYCLIC $c(x_1 \dots x_n)$?

<u>FALSE</u> FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL
CSP($\mathbb{A}$) $\in$ P

- DOES Pol($\mathbb{A}$) HAVE CYCLIC $c(x_1 \cdots x_n)$?

  PICK $a_1, \dots, a_n$ DISTINCT

$$\begin{array}{lccccccl} c(& a_1 & \cdots & & & a_n &) = & a \\ & | & | & | & | & | & & \Vert \\ c(& a_2 & \cdots & & a_{n-1} & a_1 &) = & a \end{array} \qquad \lightning$$

- $s(a,r,e,a) = s(r,a,r,e)$? $\lightning$
- $s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$ $\lightning$
- ANY SINGLE NON-TRIVIAL H1-EQUATION

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

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LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$

ω-CATEGORICAL
$CSP(\mathbb{A}) \in P$

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EXERCISE

- HOW ABOUT WNU?

  $w(x \cdots xy) = \cdots = w(yx \cdots x)$
- WHY DOES K_ω NOT PP-CONSTRUCT K_3?

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

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LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

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EXAMPLE

$NAE_n = \{(a_1 \cdots a_n) \mid a_i \text{ NOT ALL EQUAL}\}$

$\mathbb{A} = (\mathbb{N}, NAE_n)$ ω-CATEGORICAL,

$CSP(\mathbb{A})$ TRIVIAL

$$a \neq b \quad \begin{array}{l} w(a, \cdots a, b) = \\ \vdots \\ w(b a \cdots a) = \end{array} \qquad \lightning$$

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL
$CSP(\mathbb{A}) \in P$

- DOES $Pol(\mathbb{A})$ HAVE CYCLIC $c(x_1 \cdots x_n)$?
  PICK $a_1, \dots, a_n$ DISTINCT

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↯

- $s(a,r,e,a) = s(r,a,r,e)$? ↯
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$NAE_n = \{(a_1 \cdots a_n) \mid a_i \text{ NOT ALL EQUAL}\}$
$\mathbb{A} = (\mathbb{N}, NAE_n)$ ω-CATEGORICAL,
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$$a \neq b \quad \begin{array}{l} w(a, \dots, a, b) = \\ \vdots \\ w(b, a \cdots a) = \end{array}$$

↯

EXAMPLE

LET $\mathcal{C} := \{f : A^n \to A \mid n \geq 1,\ f \text{ INJECTIVE UP TO DUMMY VARIABLES}\}$

- $\mathcal{C} = Pol\overbrace{(A; \neq, x=y \to u=v)}^{=:\ \mathbb{A}}$ **EXERCISE**
- DOES NOT SATISFY ANY H1-EQUATIONS!
- $CSP(\mathbb{A}) \in P$
- SATISFIES $u \circ f(x,y) = v \circ f(y,x)$
  "PSEUDO-COMMUTATIVE"

PSEUDO-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n} :

$$f \sim g \Leftrightarrow g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(\mathbb{A})\}}$$

$$\Leftrightarrow \forall F \subseteq A \text{ FINITE}$$

$$\exists \alpha \in \mathrm{Aut}\,\mathbb{A}$$

$$g|_F = \alpha f|_F$$

" f, g ARE EQUAL MODULO ORBITS"

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EXERCISE $A^{A^n}/\sim \; =: A^{A^n}/\mathrm{Aut}\, A$

COMPACT (POLISH)

PSEUDO-EQUATIONS

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COMPACT (POLISH)

IDEA: TRY TO MIMIC PROOFS FOR FINITE STRUCTURES IN CASE OF

$\mathrm{Pol}(\mathbb{A})/_{\mathrm{Aut}\,\mathbb{A}}$

PSEUDO-EQUATIONS

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IDEA: TRY TO MIMIC PROOFS FOR FINITE STRUCTURES IN CASE OF

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FACT $\mathrm{Pol}(A)$ SATISFIES EQUATION MODULO $\mathrm{Aut}(A)$

$\Rightarrow$ $\mathrm{Pol}(A)$ SATISFIES CORRESPONDING PSEUDO-EQUATION

PSEUDO-EQUATIONS

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EXAMPLE

$$f(g(x,y),h(z)) \sim h(f(z,x),g(y,y))$$

$$\Rightarrow \exists u,v \in \overline{\mathrm{Aut}(A)}$$

$$u \circ f(g(x,y),h(z)) = v \circ h(f(z,x),g(y,y))$$

PSEUDO-EQUATIONS

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THEOREM (BARTO+P. '17)

$\mathbb{A}$ ω-CATEGORICAL, $\mathrm{Pol}(\mathbb{A})$ NON-TRIVIAL:

$$\mathrm{Pol}(\mathbb{A}) \not\xrightarrow[\text{MINION HOMOMORPHISM}]{\text{u.c.}} \mathrm{Pol}(\mathbb{K}_3)$$

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ HAS PSEUDO-SIGGERS OPERATION.

$$u \circ s(x,y,x,z,y,z) = v \circ s(y,x,z,x,z,y)$$

PSEUDO-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n}:

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IDEA: TRY TO MIMIC PROOFS FOR FINITE STRUCTURES IN CASE OF

$\mathrm{Pol}(\mathbb{A})/\mathrm{Aut}\,\mathbb{A}$

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EXAMPLE

$$f(g(x,y),h(z)) \sim h(f(z,x),g(y,y))$$

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THEOREM (BARTO+P. '17)

$\mathbb{A}$ ω-CATEGORICAL, $\mathrm{Pol}(\mathbb{A})$ NON-TRIVIAL:

$$\mathrm{Pol}(\mathbb{A}) \cancel{\xrightarrow[\text{MINION}]{u.c.}} \mathrm{Pol}(K_3)$$

HOMOMORPHISM

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ HAS PSEUDO-SIGGERS OPERATION.

$$u \circ s(x,y,x,z,y,z) = v \circ s(y,x,z,x,z,y)$$

CONVERSE TRUE IF (BARTO+KOMPATSCHER+OLŠÁK+PHAM+P. '18)

- $\mathbb{A}$ CORE"
- $\mathbb{A}$ "GIVEN BY FINITELY MANY FORBIDDEN STR

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \rightarrow u=v)$: ω-CAT, CSP $\in$ P

NO PSEUDO-CYCLIC POLYMORPHISM :

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \rightarrow u=v)$: ω-CAT, CSP ∈ P

NO PSEUDO-CYCLIC POLYMORPHISM :

- ALL POLYMORPHISMS INJECTIVE
  (UP TO DUMMY VARIABLES)

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \rightarrow u=v)$: ω-CAT, CSP ∈ P

NO PSEUDO-CYCLIC POLYMORPHISM :

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THEOREM (BARTO + BODOR + KOZIK + MOTTET + P. '23)

$\exists \mathbb{A}$ ω-CATEGORICAL ("CORE")

- $\mathrm{Pol}(\mathbb{A}) \cancel{\xrightarrow[\text{MINION HOM.}]{\text{UC}}} \mathrm{Pol}(K_3)$
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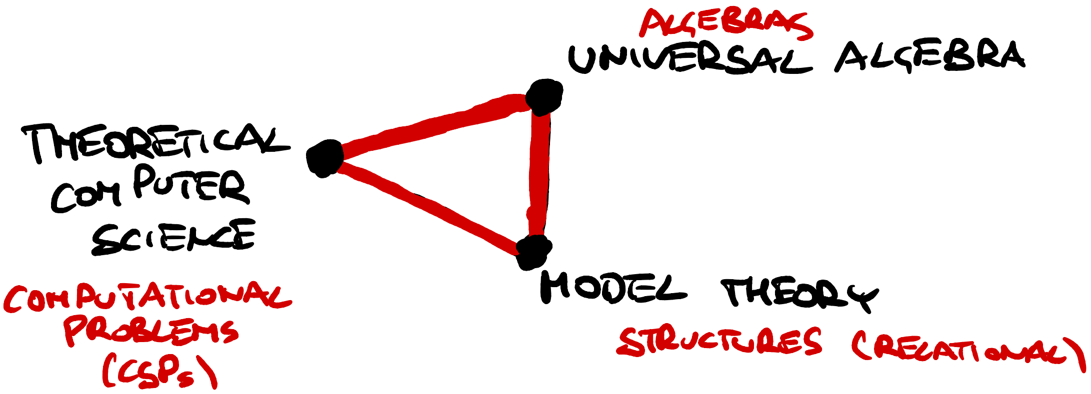
OPEN PROBLEM

- PSEUDO-4-ARY SIGGERS
- MINION ⇝ CLONE HOMOMORPHISM
- PSEUDO-WNU FOR $\mathbb{A}$ GIVEN BY FINITELY MANY FORBIDDEN STRUCTURES



PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs

ALGEBRAS

RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.

ALGEBRAS

RELATIONAL STRUCTURES

EXERCISES

- FIND A COMPLETE METRIC THAT INDUCES THE TOPOLOGY OF POINTWISE CONVERGENCE

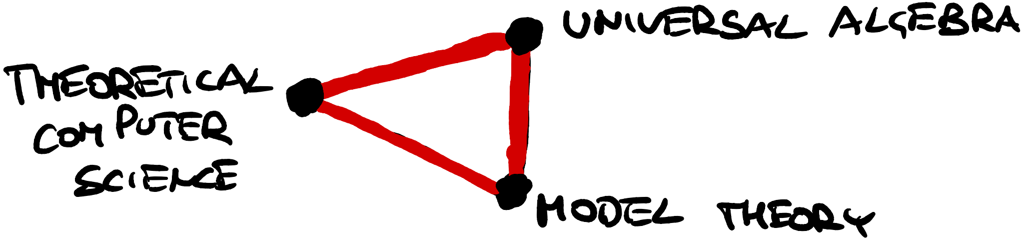
- LET $\mathbb{A}$ BE ω-CATEGORICAL

  $\forall$ FINITE $F \subseteq A$ $\exists f_F, g_F, h_F \in \mathrm{Pol}(\mathbb{A})$:

  $$f_F(g_F(x,y), h_F(y,x)) = f_F(g_F(y,x), h_F(y,x))$$

  $\Longrightarrow \exists f, g, h \in \mathrm{Pol}(\mathbb{A})$

  $$f(g(x,y), h(y,x)) = f(g(y,x), h(y,x))$$


THEORETICAL
COMPUTER
SCIENCE
UNIVERSAL ALGEBRA
MODEL THEORY

Thank you!