Constraint Satisfaction Problems

An Algebraic Approach to Classifying Computational Complexity

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Outline

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1. Introduction to CSPs
2. Tools for classifying complexity
3. Infinite-domain CSPs
4. Valued CSPs

Computational problems

- **3-SAT**

  Input: a propositional formula ϕ in 3-CNF,
  e.g. $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_2 \vee x_4) \wedge \dots$

  Output: Is ϕ satisfiable?

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problems in P = class of effectively solvable problems
NP-complete problems = problems with effectively verifiable solution; believed to be hard to solve

Constraint Satisfaction Problem

(relational) structure $\mathfrak{B} = (B; R^{\mathfrak{B}} : R \in \tau)$; finite signature τ

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Example (3-SAT):
$\mathfrak{B} = (\{0,1\}; R_{000}, R_{001}, R_{011}, R_{111})$, where $R_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$
Rewrite input $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_2 \vee x_4) \wedge \ldots$ as

$$\exists x_1, x_2, \ldots R_{001}(x_1, x_3, x_2) \wedge R_{011}(x_4, x_3, x_2) \wedge \ldots$$

Then CSP($\mathfrak{B}$) is the same problem as 3-SAT.

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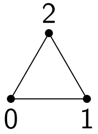
$\mathfrak{B} = K_3$ (complete graph on 3 vertices)

$G \to \mathfrak{B}$ iff G is 3-colorable.



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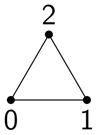

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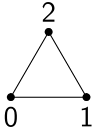

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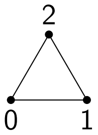

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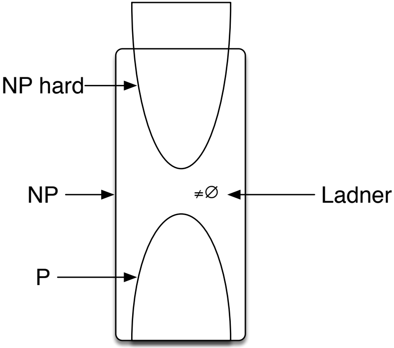
Observation: Cannot be modelled over a finite template.

Complexity of CSPs

Conjecture (Feder, Vardi '93), now theorem:

Theorem (Bulatov ('17); Zhuk ('17))

For every finite $\mathfrak{B}$, CSP($\mathfrak{B}$) is in P or NP-complete.



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Primitive positive definitions

pp-define = define by a primitive positive formula

Example: The structure $(\{0,1\}; R_{000}, R_{001}, R_{011}, R_{111})$ pp-defines the relation $XOR = \{(0,1),(1,0)\}$ by

$$R_{000}(x,y,y) \wedge R_{111}(x,y,y).$$

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If $\mathfrak{B}$ pp-defines a relation R, then $\mathrm{CSP}(\mathfrak{B}, R)$ *reduces to* $\mathrm{CSP}(\mathfrak{B})$ *in poly-time.*

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Question: How to certify that a relation is not pp-definable?

Polymorphisms

Definition (polymorphism)

An operation $f : B^n \to B$ is a polymorphism of (or preserves) $\mathfrak{B}$ if for every relation R of $\mathfrak{B}$ and for all tuples $\bar{r}_1, \ldots, \bar{r}_n \in R$ also $f(\bar{r}_1, \ldots, \bar{r}_n) \in R$ (computed row-wise).
Pol($\mathfrak{B}$) – the set of all polymorphisms of $\mathfrak{B}$

Example: The operation min is a polymorphism of $(\mathbb{Q}; <)$.

$$\begin{pmatrix} 1 \\ \wedge \\ 5 \end{pmatrix} \begin{pmatrix} 2 \\ \wedge \\ 3 \end{pmatrix} \begin{matrix} \overset{\min}{\rightarrow} \\ \\ \overset{\min}{\rightarrow} \end{matrix} \begin{pmatrix} 1 \\ \wedge \\ 3 \end{pmatrix}$$

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Example (projections): For every structure $\mathfrak{B}$, $n \in N$ and $i \in \{1, \ldots, n\}$, $\pi_i^n : B^n \to B$ defined by

$$\pi_i^n(x_1, \ldots, x_n) = x_i$$

is a polymorphism of $\mathfrak{B}$.

Use of polymorphisms

1. Certify that a relation is not pp-definable

Theorem (Bodnarčuk, Kalužnin, Kotov, Romov ('69); Geiger ('68))

$\mathfrak{B}$, $\mathfrak{B}'$ - *structures on the same finite domain*
All relations of $\mathfrak{B}'$ *are pp-definable in* $\mathfrak{B}$ *iff* $\mathrm{Pol}(\mathfrak{B}) \subseteq \mathrm{Pol}(\mathfrak{B}')$.

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$\mathfrak{A}$ – input for $\mathrm{CSP}(\mathfrak{B})$:
If $R^{\mathfrak{A}} \neq \emptyset$ and $R^{\mathfrak{B}} = \emptyset$ for some R, then $\mathfrak{A} \not\to \mathfrak{B}$.
Otherwise, $a \mapsto c, a \in A$ is a homomorphism $\mathfrak{A} \to \mathfrak{B}$.

Pp-constructions

pp-power of $\mathfrak{B}$: a σ-structure $\mathfrak{C} = (B^d; R^{\mathfrak{C}} : R \in \sigma)$ for some $d \in \mathbb{N}$ where $R^{\mathfrak{C}} \subseteq B^{dk}$ is pp-definable in $\mathfrak{B}$ for every $R \in \sigma$

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Algebraic description of pp-constructability

height-one (h1) identity: equation of the form

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_m \; f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$$

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Theorem (Barto, Opršal, Pinsker ('15))

$\mathfrak{B}$, $\mathfrak{B}'$ – *finite structures*
$\mathfrak{B}$ *pp-constructs* $\mathfrak{B}'$ *iff* $\mathrm{Pol}(\mathfrak{B}')$ *satisfies every h1-identity satisfied in* $\mathrm{Pol}(\mathfrak{B})$*.*

Finite-domain CSP dichotomy theorem

Theorem (Bulatov ('17); Zhuk ('17))

If $\mathfrak{B}$ is a finite structure, then precisely one of the following holds:

- $\mathfrak{B}$ *pp-constructs* K_3 *and* $\mathrm{CSP}(\mathfrak{B})$ *is NP-complete.*
- $\mathfrak{B}$ *has a cyclic polymorphism* f *of some arity* n*, i.e.,* f *satisfying*

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$$\forall x, y, z \;\; s(x, y, z, x) = s(y, x, y, z)$$

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Observation: It is decidable which of the two cases applies.

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Fact: Pol(K_3) satisfies the same h1-identities as the projections on $\{0, 1\}$.

Corollary: First item is equivalent to 'Pol($\mathfrak{B}$) satisfies only the h1-identities satisfied by projections on $\{0, 1\}$'.

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If $\mathrm{Aut}(\mathfrak{B})$ *is oligomorphic,* $\mathfrak{B}$ *pp-constructs* K_3 *iff* $\mathrm{Pol}(\mathfrak{B})$ *satisfies only the h1-identities satisfied by projections on* $\{0, 1\}$.

Infinite-domain dichotomy conjecture

Definition

- $\mathfrak{B}$ is finitely bounded if there exists a universal sentence ϕ such that a finite structure $\mathfrak{A}$ embeds in $\mathfrak{B}$ iff $\mathfrak{A} \models \phi$.
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Verified for structures fo-definable in: $(\mathbb{Q}, <)$, any homogeneous graph, unary ω-categorical structures, ...

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Constraint satisfaction variants

$\mathfrak{B}$ – fixed relational structure

Input: list of constraints (e.g. as a pp-formula)

Output:

- **CSP**: Decide whether there is a solution that satisfies all constraints.
- **MaxCSP**: Find the maximal number of constraints that can be satisfied at once.
- **VCSP**: Find the minimal cost with which the constraints can be satisfied (each constraint comes with a cost depending on the chosen values).

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Observation: VCSP generalizes CSP and MaxCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost 1 (for MaxCSP) or ∞ (for CSP).

VCSP

A valued structure Γ consists of:

- (countable) domain D
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k, a function $R^\Gamma \colon D^k \to \mathbb{Q} \cup \{\infty\}$

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \dots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Question: Is

$$\inf_{\bar{a} \in D^n} \phi(\bar{a}) \leq u \text{ in } \Gamma?$$

Directed Max-Cut as a VCSP

Example:

Input: $G = (V, E)$ – finite directed graph

Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.

Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ_{MC} be a valued structure where:

- $D = \{0, 1\}$
- $\tau = \{R\}$, R binary

$$R(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

Directed Max-Cut as a VCSP

Example:

Input: $G = (V, E)$ – finite directed graph

Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.

Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Take vertices of G as variables. The size of a maximal cut of G is

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every instance of VCSP(Γ_{MC}) corresponds to a digraph

$\rightsquigarrow$ VCSP(Γ_{MC}) is the Directed Max-Cut problem (NP-complete)

Pp-constructions for VCSPs

- pp-definitions can be generalized to valued structures (e.g. $\wedge \rightsquigarrow +$, $\exists \rightsquigarrow \inf$, and more operators)
- we can define a notion of a pp-construction

Proposition (Bodirsky, Lutz, S.)

If $\mathrm{Aut}(\Gamma)$ *and* $\mathrm{Aut}(\Delta)$ *are oligomorphic and* Γ *pp-constructs* Δ*, then* $\mathrm{VCSP}(\Delta)$ *reduces to* $\mathrm{VCSP}(\Gamma)$ *in poly-time.*

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Corollary

If Aut(Γ) *is oligomorphic and* Γ *pp-constructs* K_3, *then* VCSP(Γ) *is NP-hard.*

Fractional polymorphisms

Definition (fractional polymorphism)

Γ – valued τ-structure with domain D

A fractional polymorphism of Γ of arity n is a probablity distribution ω on operations $D^n \to D$ such that for every k-ary $R \in \tau$ and $a^1, \dots, a^n \in D^k$

$$\underbrace{E_\omega[f \mapsto R(f(a^1, \dots, a^n))]}_{\text{expected value}} \leq \underbrace{\frac{1}{n} \sum_{j=1}^{n} R(a^j)}_{\text{arithmetic mean}} .$$

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Example: For every Γ and $n \in \mathbb{N}$, ω defined by

$$\omega(\pi_i^n) = \frac{1}{n} \text{ for every } i \in \{1, \ldots, n\}$$

is a fractional polymorphism of Γ.

Tractability for VCSPs

Known for finite-domain VCSPs:

Theorem (adapted from Kozik, Ochremiak ('15) and Kolmogorov, Krokhin, Rolínek ('15))

If Γ is a finite valued structure, then precisely one of the following holds:

- Γ *pp-constructs* K_3 *and* VCSP(Γ) *is NP-complete.*
- Γ *has a cyclic fractional polymorphism and* VCSP(Γ) *is in P.*

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Theorem (Bodirsky, Lutz, S.)

Let $\mathfrak{B}$ be a finitely bounded homogeneous structure such that $\text{Aut}(\Gamma) = \text{Aut}(\mathfrak{B})$*. If Γ has a canonical pseudo cyclic fractional polymorphism, then* VCSP(Γ) *is in P.*

Application of infinite-domain VCSPs

Definition (Resilience)

q – fixed conjunctive query (pp-formula)
Input: a finite database $\mathfrak{A}$ (relational structure)
Output: minimal number of tuples to be removed from relations of $\mathfrak{A}$, so that $\mathfrak{A} \not\models q$

Appears first in the paper of Meliou, Gatterbauer, Moore, Suciu ('10).

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- Can be modelled as a VCSP when considered over bag databases (each tuple appears with a multiplicity $m \in \mathbb{N}$).
- All queries that contain a cycle require infinite-domain valued structures as templates.
- Enables systematic study of resilience problems.

Thank you for your attention

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