Identifying Tractable Quantified Temporal Constraints within Ord-Horn

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(Quantified) constraint satisfaction problem

(relational) structure $\mathfrak{B} = (B; R^{\mathfrak{B}} : R \in \tau)$; finite signature τ

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Constraint Satisfaction Problem for $\mathfrak{B}$ (CSP($\mathfrak{B}$)):

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Intuition:

- UP: tries to force $u = v$ for some u, v with $[\![u]\!] \neq [\![v]\!]$
- EP: obeys the constraints, does not introduce unnecessary equalities

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Temporal (Q)CSPs (relations fo-definable in $(\mathbb{Q}; <)$):

- classification of CSPs (Bodirsky, Kára '10)
- some classification results on QCSPs (Charatonik, Wrona '08; Chen, Wrona '12; Bodirsky, Chen, Wrona '14; Wrona '14)

Ord-Horn constraints

Ord-Horn (OH) fragment: temporal structures whose relations are definable by an OH formula, i.e., a conjunction of clauses of the form

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$$\mathrm{M}^+ := \{(x, y, z) \in \mathbb{Q}^3 \mid x = y \Rightarrow x \geq z\}$$
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Theorem (Wrona '14)

Let $\mathfrak{B}$ be an OH structure. Then one of the following holds:

- $\mathfrak{B}$ *is guarded OH.*
- QCSP($\mathfrak{B}$) *is* coNP-*hard.*
- $\mathfrak{B}$ *pp-defines* M^+ *or* M^-*.*

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Let $\mathfrak{A}, \mathfrak{B}$ be structures with the same domain. If every relation of $\mathfrak{B}$ is qpp-definable in $\mathfrak{A}$, then $\mathrm{QCSP}(\mathfrak{B})$ *reduces to* $\mathrm{QCSP}(\mathfrak{A})$ *in* PTIME.

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Complexity of QCSP($\mathbb{Q}; \mathrm{M}^+$): left open in [Bodirsky, Chen, Wrona '14]
$\hookrightarrow$ could have been anywhere between PTIME and PSPACE

Tractability of QCSP($\mathbb{Q}; \mathrm{M}^+$)

Theorem (Rydval, S., Wrona '24)

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We write $A \prec B$ meaning $x \prec y$, $\forall x \in A, y \in B$.

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Fact: It is possible to pp-define from M^+ constraints of the form

$$\left(\bigwedge_{v \in A} x = v\right) \Rightarrow x \geq z.$$

Sketch of the algorithm

- expand ϕ by constraints ψ of the form

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- accept if no new constraints can be derived

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For $x, z \in \mathrm{V}$:

$$x\text{-}z\text{-cut} := \{u \in \mathrm{V}_\forall \mid (\mathrm{V}_\exists \cap \{x, z\}) \prec u\} \setminus \{z\}$$

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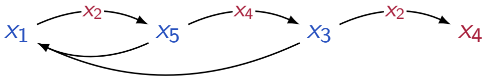
Example: $\Phi := \exists u \forall v \exists w \forall x \forall y\ \phi(u, v, w, x, y)$

- u-w-cut $= \{x, y\}$;
- u-x-cut $= \{v, y\}$;
- v-x-cut $= \{v, y\}$.

Example of the run of the algorithm

$$\Phi = \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \big((x_1 = x_2 \Rightarrow x_1 \geq x_5) \wedge (x_3 = x_2 \Rightarrow x_3 \geq x_4) \wedge (x_5 = x_4 \Rightarrow x_5 \geq x_3) \wedge (x_3 \geq x_1) \wedge (x_5 \geq x_1) \big).$$



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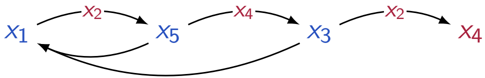

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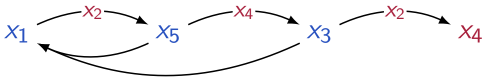

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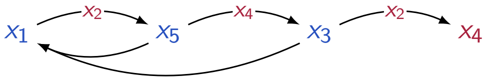

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- x_1-x_3-cut $= \{x_4\}$ $\rightsquigarrow$ $\uparrow_{x_2} \setminus (\{x_1, x_3\} \cup x_1\text{-}x_3\text{-cut}) = \{x_2\}$

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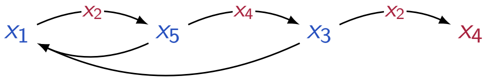

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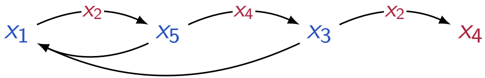

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- Hence, the algorithm expands ϕ by $(x_1 = x_2 \Rightarrow x_1 \geq x_3)$.
- Now $\phi \wedge (x_1 = x_2)$ implies $x_1 = x_2 = x_3 \geq x_4$.

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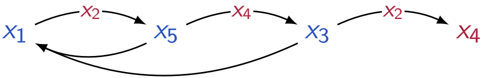

Claim: The algorithm derives $(x_1 \geq x_4)$, and thereby rejects on Φ.

- $\phi \wedge (x_1 = x_2) \wedge (x_1 = x_4)$ implies $x_1 = x_2 = x_4 = x_5 = x_3$.
- Hence, $\phi \wedge (\bigwedge_{v \in \uparrow_{x_2} \setminus \{x_1, x_3\}} x_1 = v) \wedge (x_1 < x_3)$ is not satisfiable.
- x_1-x_3-cut $= \{x_4\}$ $\quad \rightsquigarrow \quad$ $\uparrow_{x_2} \setminus (\{x_1, x_3\} \cup x_1\text{-}x_3\text{-cut}) = \{x_2\}$
- Hence, the algorithm expands ϕ by $(x_1 = x_2 \Rightarrow x_1 \geq x_3)$.
- Now $\phi \wedge (x_1 = x_2)$ implies $x_1 = x_2 = x_3 \geq x_4$.
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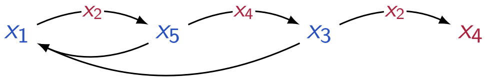

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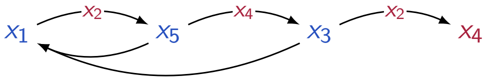

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Tractability consequences

Corollary

QCSP($\mathfrak{B}$) is in PTIME *if $\mathfrak{B}$ is a structure whose relations are* *definable* *by a conjunction of clauses of the form*

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Equivalently: structures $\mathfrak{B}$ whose relations lie both in the OH fragment and the $\pi\pi$-fragment (preserved by the operation $\pi\pi$ – 'projection-projection' operation from [Bodirsky, Kára '09]).

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Lemma (Rydval, S., Wrona '24)

Let $\mathfrak{B}$ be an OH structure that is not contained in the $\pi\pi$ fragment and pp-defines M^+. Then QCSP($\mathfrak{B}$) *is* coNP*-hard.*

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Theorem (Rydval, S., Wrona '24)

Let $\mathfrak{B}$ be an OH structure. Then QCSP($\mathfrak{B}$) *is in* PTIME *if $\mathfrak{B}$ is guarded OH, contained in the $\pi\pi$ fragment, or in the dual $\pi\pi$ fragment. Otherwise,* QCSP($\mathfrak{B}$) *is* coNP*-hard.*

Open questions

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Question 2: Is QCSP($\mathbb{Q}; x \neq y \vee x \geq z \vee x > w$) in PTIME?

Answer 'yes' to Question 2 $\Rightarrow$ tractability for QCSP($\mathfrak{B}$) for all $\mathfrak{B}$ contained in the *mi* fragment (preserved by the operation *mi* [Bodirsky, Kára '09])

Thank you for your attention

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