Identifying Tractable Quantified Temporal Constraints

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(Quantified) constraint satisfaction problem

(relational) structure $\mathfrak{B} = (B; R^{\mathfrak{B}} : R \in \tau)$; finite signature τ

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Intuition:

- UP: tries to force $u = v$ for some u, v with $[\![u]\!] \neq [\![v]\!]$
- EP: obeys the constraints, does not introduce unnecessary equalities

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Temporal (Q)CSPs (relations fo-definable in $(\mathbb{Q}; <)$):

- classification of CSPs (Bodirsky, Kára '10)
- some classification results on QCSPs (Charatonik, Wrona '08; Chen, Wrona '12; Bodirsky, Chen, Wrona '14; Wrona '14)

Ord-Horn constraints

Ord-Horn (OH) fragment: temporal structures whose relations are definable by an OH formula, i.e., a conjunction of clauses of the form

$$(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee x_{k+1} \geq y_{k+1}) \text{ (last disjunct is optional).}$$

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QCSP($\mathbb{Q}$; R) where R is defined by $(x_1 \neq x_2 \vee x_3 \geq x_4) \wedge \phi$ is:

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- PSPACE-complete if ϕ is empty (Zhuk, Martin, Wrona '23)

Three cases for Ord-Horn constraints

Theorem (Wrona '14)

Let $\mathfrak{B}$ be an OH structure. Then one of the following holds:

- $\mathfrak{B}$ *is guarded OH.*
- QCSP($\mathfrak{B}$) *is* coNP*-hard.*
- $\mathfrak{B}$ *pp-defines* M^{+} *or* M^{-}*.*

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$$\mathrm{M}^+ \coloneqq \{(x, y, z) \in \mathbb{Q}^3 \mid x = y \Rightarrow x \geq z\}$$
$$\mathrm{M}^- \coloneqq \{(x, y, z) \in \mathbb{Q}^3 \mid x = y \Rightarrow x \leq z\}$$

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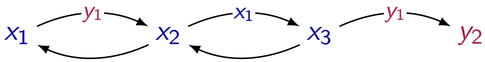
Complexity of QCSP($\mathbb{Q}; \mathrm{M}^+$): left open in [Bodirsky, Chen, Wrona '14]
$\hookrightarrow$ could have been anywhere between PTIME and PSPACE

Example instance of QCSP($\mathbb{Q}; \mathrm{M}^+$)

$$\Phi = \exists x_1 \forall y_1 \exists x_2 \forall y_2 \exists x_3 \big((x_1 = y_1 \Rightarrow x_1 \geq x_2) \wedge (x_2 = x_1 \Rightarrow x_2 \geq x_3) \\ \wedge (x_3 = y_1 \Rightarrow x_3 \geq y_2) \wedge (x_3 \geq x_2) \wedge (x_2 \geq x_1) \big).$$



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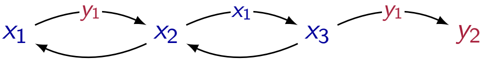

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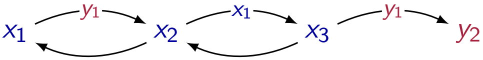

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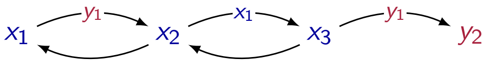

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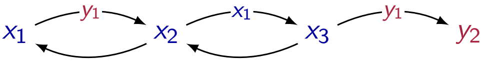

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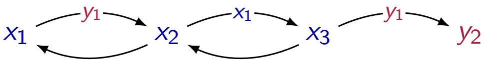

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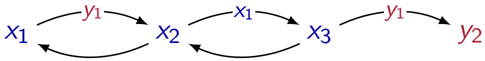

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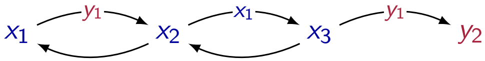

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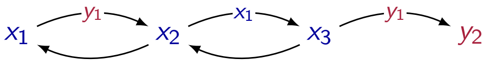

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- the UP has a winning strategy on this instance $\Rightarrow$ Φ is false

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- we write $A \prec B$ meaning $x \prec y$, $\forall x \in A, y \in B$

The setup

Wanted: PTIME-algorithm for QCSP($\mathbb{Q}; \mathrm{M}^+$)
$\mathrm{M}^+ = \{(x, y, z) \in \mathbb{Q}^3 \mid x = y \Rightarrow x \geq z\}$

- fix instance Φ of QCSP($\mathbb{Q}; \mathrm{M}^+$) over variables $\mathrm{V} = \mathrm{V}_\exists \cup \mathrm{V}_\forall$
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Fact: It is possible to pp-define from M^+ constraints of the form

$$\left(\bigwedge_{v \in A} x = v\right) \Rightarrow x \geq z$$

by definitions of linear length.

x-z-cut

For $x, z \in \mathrm{V}$:

$$x\text{-}z\text{-cut} := \{u \in \mathrm{V}_\forall \mid (\mathrm{V}_\exists \cap \{x, z\}) \prec u\} \setminus \{z\}$$

- x-z-cut comprises variables that the UP can play equal to x to trigger the constraint $x \geq z$
- z is removed so that the constraint does not become trivial

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Example: $\Phi \coloneqq \exists u \forall v \exists w \forall x \forall y \; \phi(u, v, w, x, y)$

- u-w-cut $= \{x, y\}$
- u-x-cut $= \{v, y\}$
- v-x-cut $= \{v, y\}$

Sketch of the algorithm

- expand ϕ by constraints ψ of the form

$$\left(\bigwedge_{v \in A \setminus x\text{-}z\text{-cut}} x = v\right) \Rightarrow x \geq z$$

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- accept if no new constraints can be derived

Algorithm for QCSP($\mathbb{Q}; \mathrm{M}^+$)

Input: an instance Φ of QCSP($\mathbb{Q}; \mathrm{M}^+$) with the quantifier-free part ϕ
Output: *true* or *false*

while ϕ *changes* **do**
  for $x, z, u \in \mathrm{V}$ **do**
    if ϕ *contains the clause* $(x \geq z)$ *or* $(z \geq x)$, *where* $x \prec z$ *and* $z \in \mathrm{V}_\forall$ **then**
      return *false*;
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$\hookrightarrow$ the algorithm runs in PTIME

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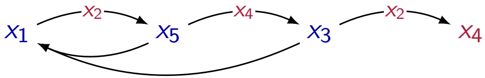
$$\Phi = \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \big((x_1 = x_2 \Rightarrow x_1 \geq x_5) \wedge (x_3 = x_2 \Rightarrow x_3 \geq x_4) \wedge (x_5 = x_4 \Rightarrow x_5 \geq x_3) \wedge (x_3 \geq x_1) \wedge (x_5 \geq x_1) \big).$$



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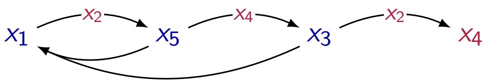

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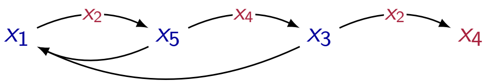

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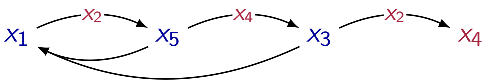

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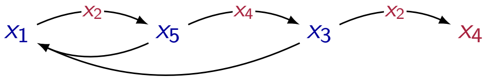

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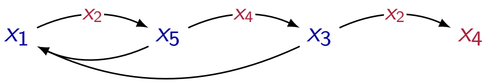

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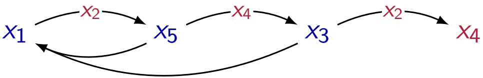

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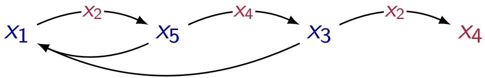

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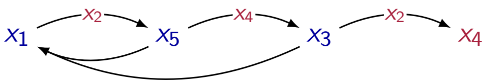

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Lemma (Rydval, S., Wrona '24)

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Whenever the algorithm rejects, it derived

$$x \geq z \text{ or } z \geq x \text{ where } x \prec z, z \in \mathrm{V}_{\forall}.$$

Lemma $\Rightarrow$ Φ is false $\Rightarrow$ the algorithm rejects false instances

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- *If the proof system does not derive $\mathcal{P}(x, z; \emptyset)$ or $\mathcal{P}(z, x; \emptyset)$ for $x \prec z$, $z \in \mathrm{V}_\forall$, then Φ is true.*

$\rightsquigarrow$ the algorithm accepts correctly

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$\hookrightarrow$ conditional constraints are necessary for this to be true

The proof system $\mathcal{P}$

"Trial version" of $\mathcal{P}$

Initialize	$\mathcal{P}(x,x;\emptyset)$:− $x \in V$
Simplify	$\mathcal{P}(x,z;A \setminus x\text{-}z\text{-cut})$:− $\mathcal{P}(x,z;A)$
Transitivity	$\mathcal{P}(x,z;A)$:− $\mathcal{P}(x,y;A) \wedge \mathcal{P}(y,z;\emptyset)$
Constraint	$\mathcal{P}(x,z;y)$:− $(x = y \Rightarrow x \geq z) \wedge y \in \mathrm{V}_{\forall}$

The proof system $\mathcal{P}$

"Trial version" of $\mathcal{P}$

Initialize	$\mathcal{P}(x,x;\emptyset)$:− $x \in V$
Simplify	$\mathcal{P}(x,z;A \setminus x\text{-}z\text{-cut})$:− $\mathcal{P}(x,z;A)$
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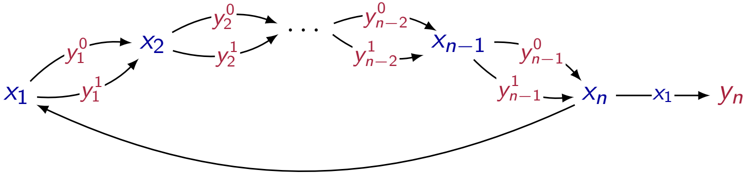
Example (transitivity): $\mathcal{P}(x,z;A)$:− $\mathcal{P}(x,y;A) \wedge \mathcal{P}(y,z;\emptyset)$

$$\left(\left(\bigwedge_{v \in A} x = v \right) \Rightarrow x \geq y \right) \wedge (y \geq z)$$
$$\rightsquigarrow \left(\bigwedge_{v \in A} x = v \right) \Rightarrow x \geq z$$

$\mathcal{P}$ does not give a PTIME-algorithm

Example: $\Phi = \exists x_1 \forall y_1^0 \forall y_1^1 \exists x_2 \forall y_2^0 \forall y_2^1 \cdots \exists x_{n-1} \forall y_{n-1}^0 \forall y_{n-1}^1 \exists x_n \forall y_n$



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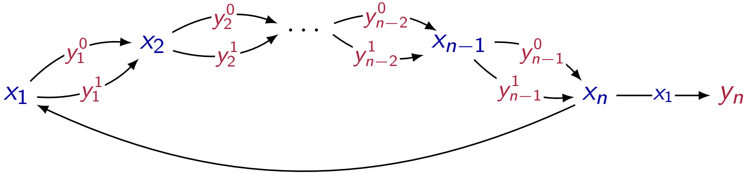

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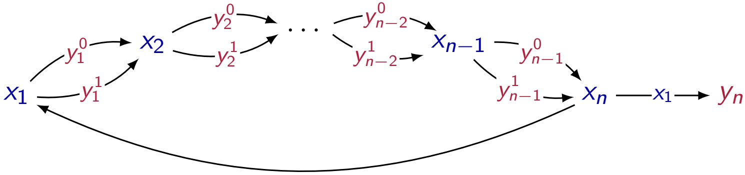

- $\mathcal{P}$ follows shortest derivation sequences
- $\mathcal{P}$ derives $\mathcal{P}(x_1, x_n; \{y_1^{i_1}, \dots, y_{n-1}^{i_{n-1}}\})$ for all $i_1, \dots, i_{n-1} \in \{0, 1\}$

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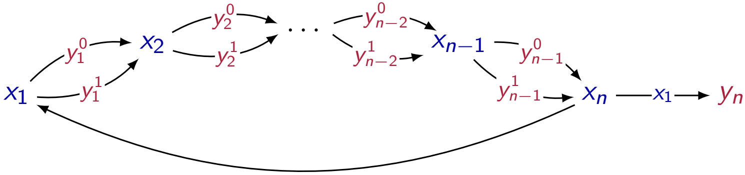

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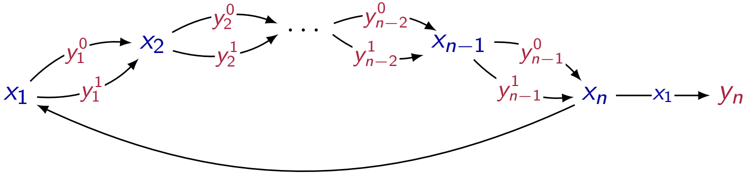

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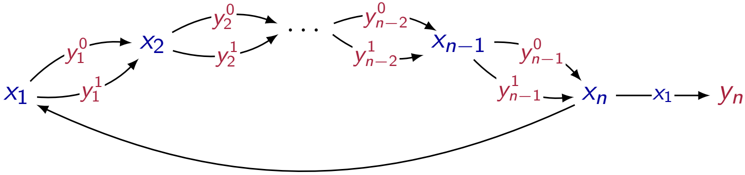

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$\mathcal{P}$ may derive exponentially many predicates
$\Rightarrow$ does not give a PTIME-algorithm

Tractability consequences

Theorem (Rydval, S., Wrona '24)

QCSP$(\mathbb{Q}; \mathrm{M}^{+})$ *is in* PTIME.

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QCSP($\mathbb{Q}; \mathrm{M}^+$) *is in* PTIME.

Corollary

QCSP($\mathfrak{B}$) *is in* PTIME *if* $\mathfrak{B}$ *is a structure whose relations are definable by a conjunction of clauses of the form*

$$(x \neq y_1 \vee \cdots \vee x \neq y_k \vee x \geq z)$$

for $k \geq 0$ *and where the last disjunct* $(x \geq z)$ *may be omitted.*

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Equivalently: structures $\mathfrak{B}$ whose relations lie both in the OH fragment and the $\pi\pi$ fragment (pp fragment from [Bodirsky, Kára '09]).

Complexity dichotomy for Ord-Horn constraints

Lemma (Rydval, S., Wrona '24)

Let $\mathfrak{B}$ be an OH structure that is not contained in the $\pi\pi$ fragment and pp-defines M^{+}. Then QCSP($\mathfrak{B}$) *is* coNP-*hard.*

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$\rightsquigarrow$ we can prove only coNP-hardness

Open questions

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$\hookrightarrow$ a maximal tractable fragment for CSPs

$\hookrightarrow$ the last such fragment where it is unknown whether it is a maximal tractable fragment for QCSPs

Thank you for your attention

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