Classification Transfer for Constraint Satisfaction Problems

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Constraint Satisfaction Problems

(relational) structure $\mathfrak{B} = (B; R^{\mathfrak{B}} : R \in \tau)$; finite signature τ

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Constraint Satisfaction Problem for $\mathfrak{B}$ (CSP($\mathfrak{B}$)):

Input: conjunction ϕ of atomic formulas

Question: Is ϕ satisfiable in $\mathfrak{B}$?

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Example (3-SAT):

$\mathfrak{B} = (\{0,1\}; R_{000}, R_{001}, R_{011}, R_{111})$, where $R_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$

Rewrite input $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_2 \vee x_4) \wedge \ldots$ as

$$R_{001}(x_1, x_3, x_2) \wedge R_{011}(x_4, x_3, x_2) \wedge \ldots$$

CSP($\mathfrak{B}$) is the same problem as 3-SAT.

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Example (graph acyclicity):

$\mathfrak{B} = (\mathbb{Q}; <) \rightsquigarrow$ digraph $(\mathbb{Q}; E)$

Write the edges of an input digraph G in a conjunction

$$E(x_1, x_2) \wedge E(x_3, x_4) \ldots$$

The formula is satisfiable in $(\mathbb{Q}; E)$ iff G has no directed cycle.

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Goal: Classify the complexity of CSP($\mathfrak{B}$) depending on $\mathfrak{B}$.

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- all finite structures (Bulatov '17; Zhuk '17)

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$\hookrightarrow$ many concrete classes where it is open

Cardinal Direction Calculus

- $\mathfrak{C} = (\mathbb{Q}^2; \mathrm{N}, \mathrm{E}, \mathrm{S}, \mathrm{W}, \mathrm{NE}, \mathrm{SE}, \mathrm{SW}, \mathrm{NW})$ (North, East, etc.)

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Conjecture: CSP($\mathfrak{B}$) is in P iff all relations of $\mathfrak{B}$ are definable by an Ord-Horn formula, i.e., a conjunction of clauses of the form

$$(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee x_{k+1} \geq y_{k+1}) \quad \text{(last disjunct is optional).}$$

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Consider structures $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$, where

$$(a_1, \ldots a_n) <_i (b_1, \ldots, b_n) \text{ iff } a_i < b_i \text{ and}$$
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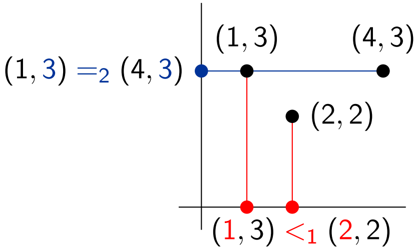
Example: $n = 2$



Plan of attack

- classify the complexity of CSP($\mathfrak{D}$) where $\mathfrak{D}$ is a fo-expansion of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$ using the results for fo-expansions of $(\mathbb{Q}; <)$

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Example (Interval Algebra): $\mathfrak{B} = (\mathbb{I}; \mathsf{s} \cup \mathsf{f})$

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Consequences:

- complexity classification for CDC$_n$ and the Block Algebra
- tractable cases are definable by Ord-Horn formulas
- solves the open problems from '99 and '02

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Verify the infinite-domain CSP dichotomy conjecture for:

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- more structures with a product structure, e.g. finite structures with $(\mathbb{Q}; <)$
- structures fo-interpretable over $(\mathbb{Q}; <)$

Thank you for your attention

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