Strong subalgebras: version 4

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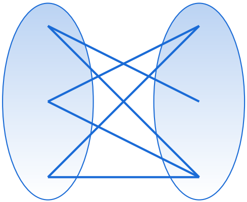

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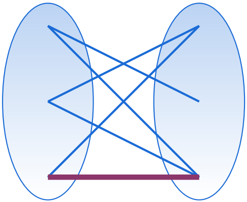

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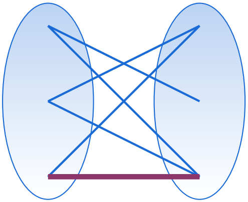

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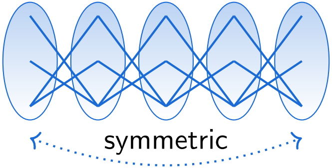

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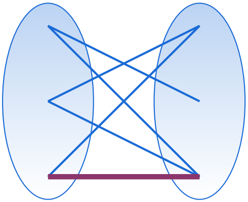

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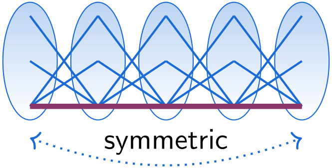

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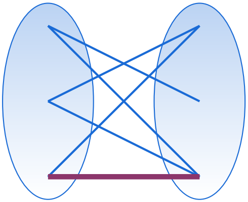

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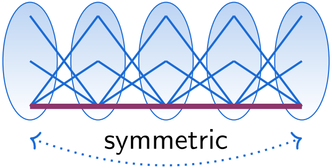

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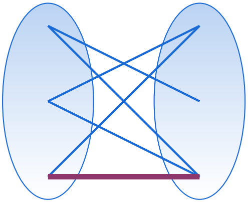

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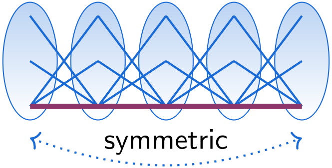

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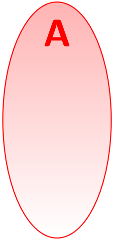

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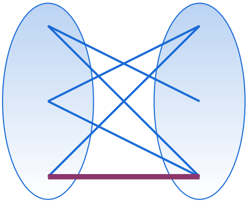

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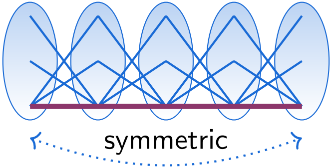

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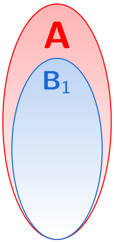

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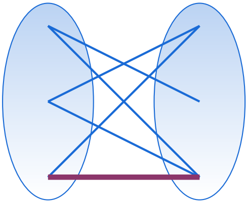

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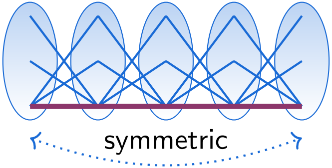

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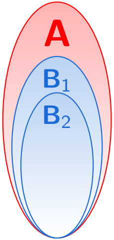

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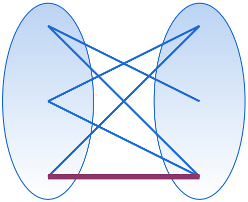

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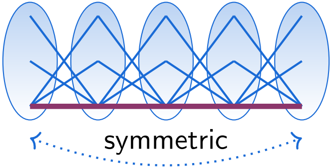

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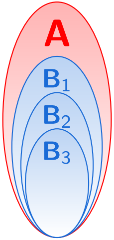

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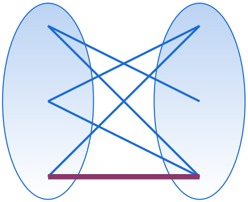

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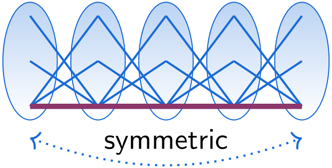

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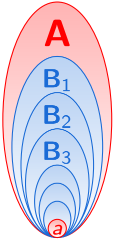

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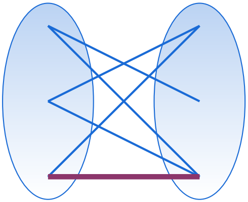

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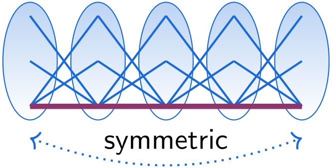

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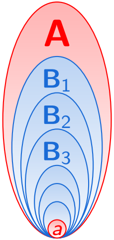

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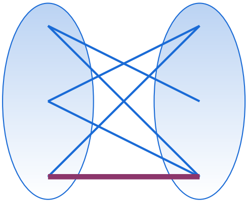

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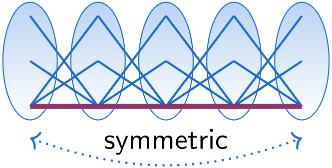

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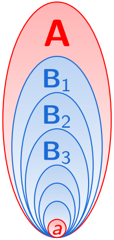

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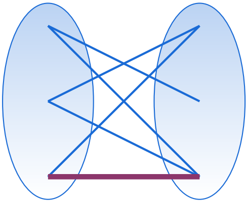

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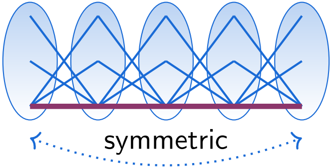

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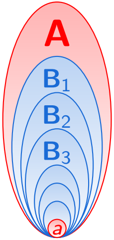

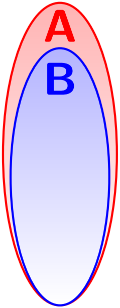

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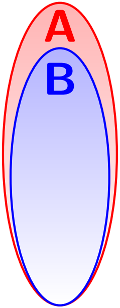

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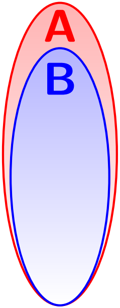

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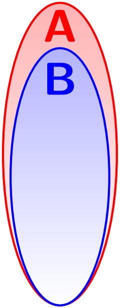

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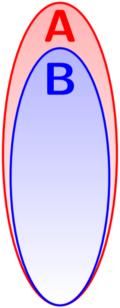

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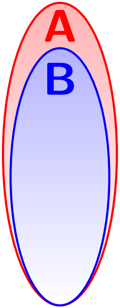

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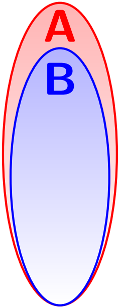

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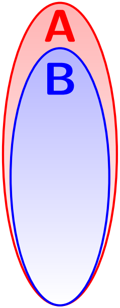
Absorbing Subalgebras (version 1)

Barto, Kozik



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Properties

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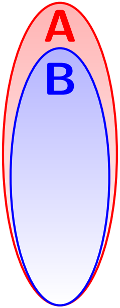

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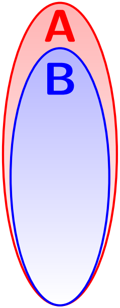
Absorbing Subalgebras (version 1)



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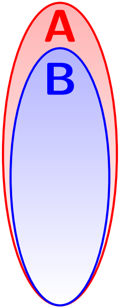
Barto, Kozik



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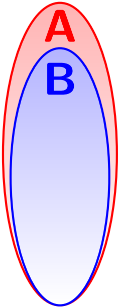

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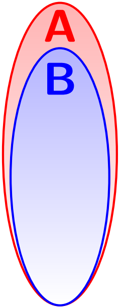

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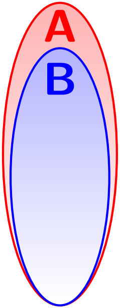

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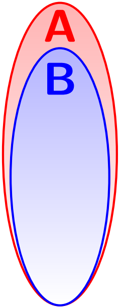

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2. Affine case is too weak and cannot be used. This works mostly for the bounded width case.

Strong Subalgebras (version 4)

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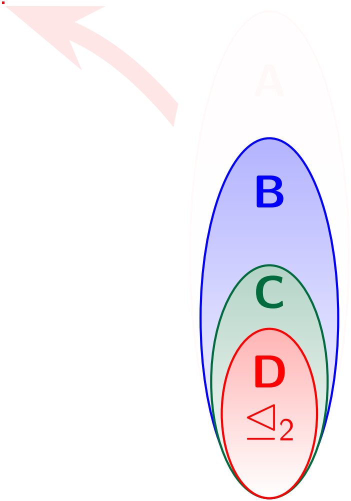

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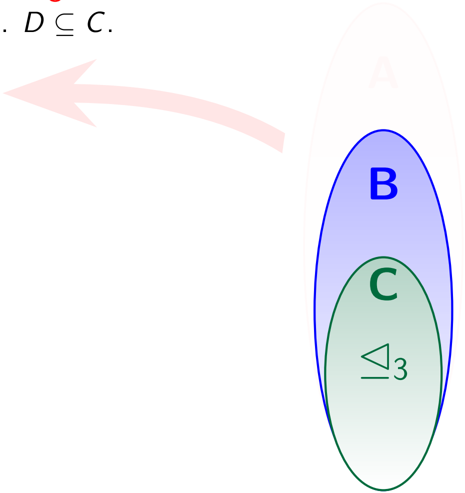

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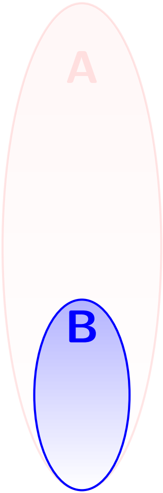

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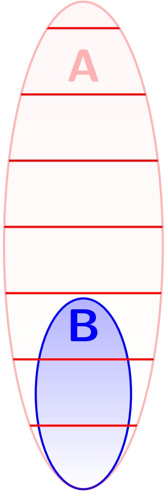

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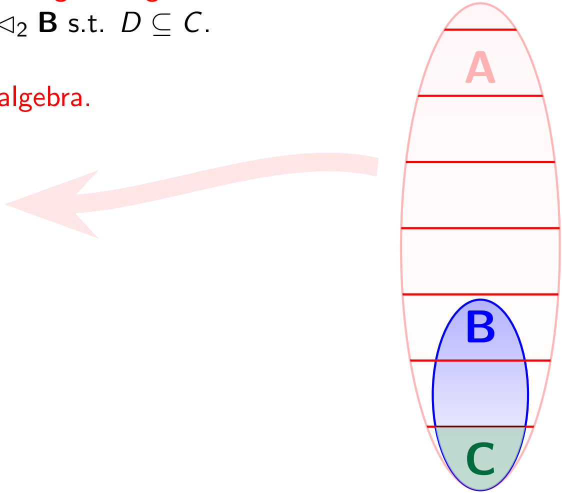

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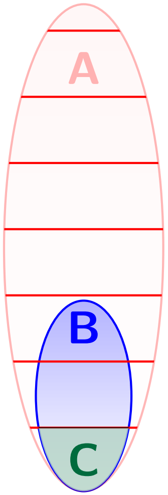

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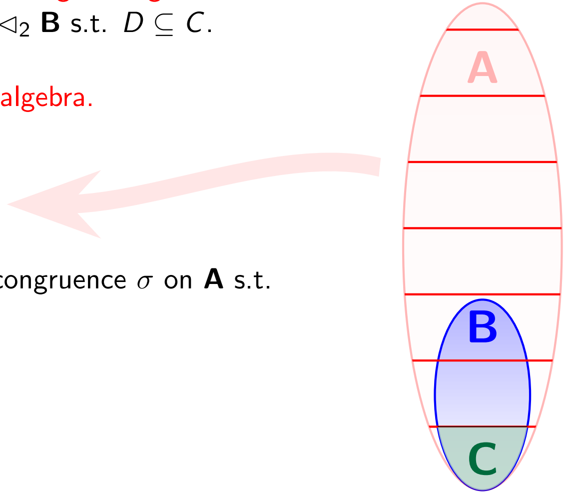

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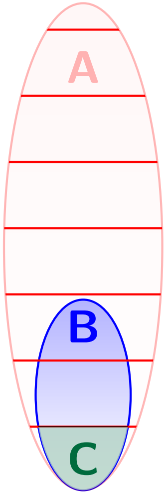

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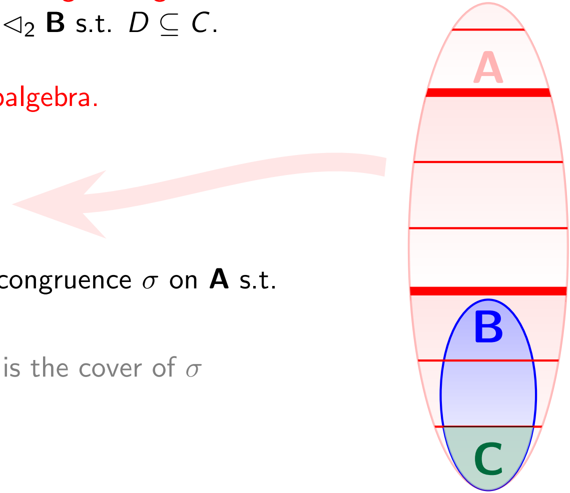

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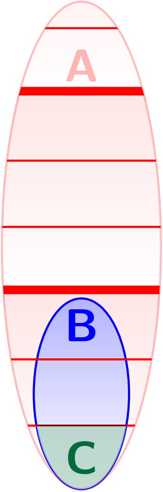

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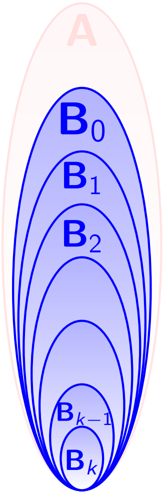

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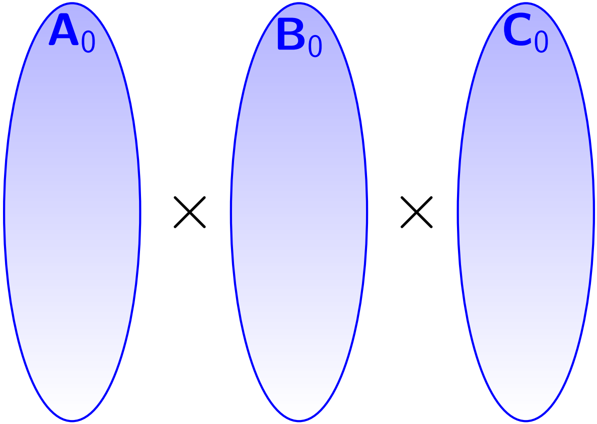

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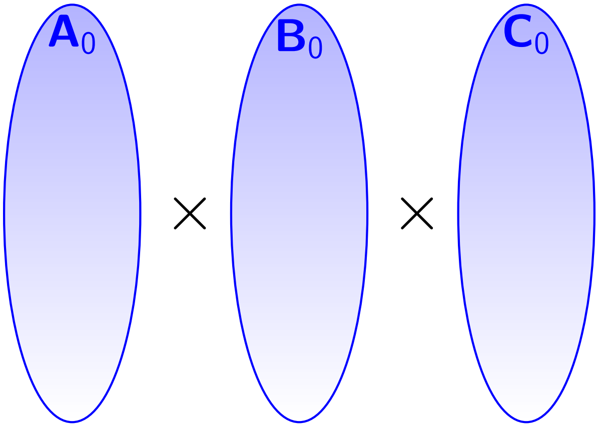

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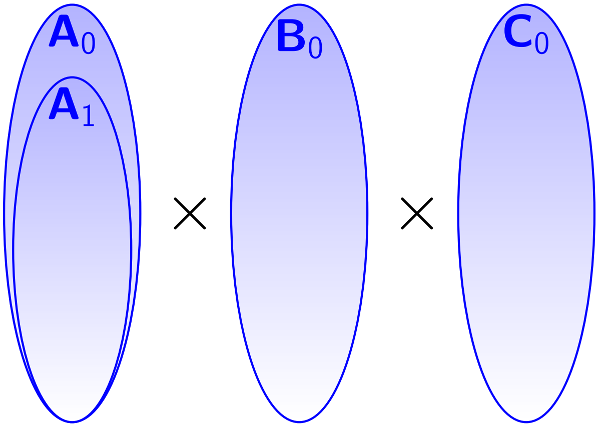

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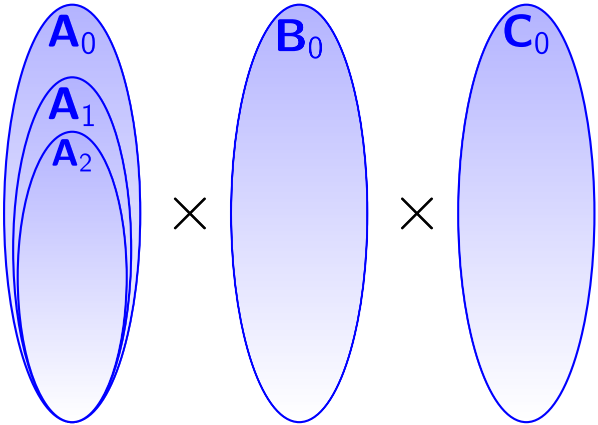

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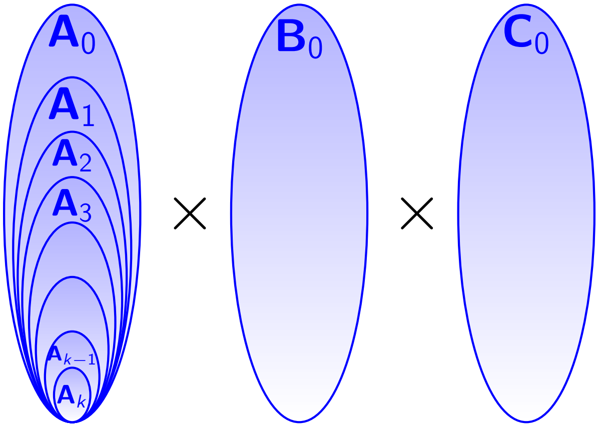

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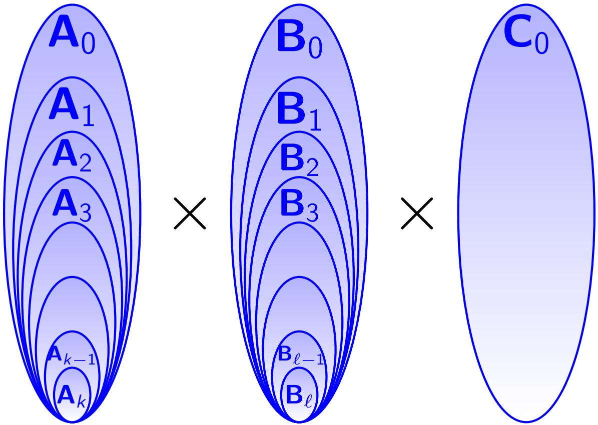

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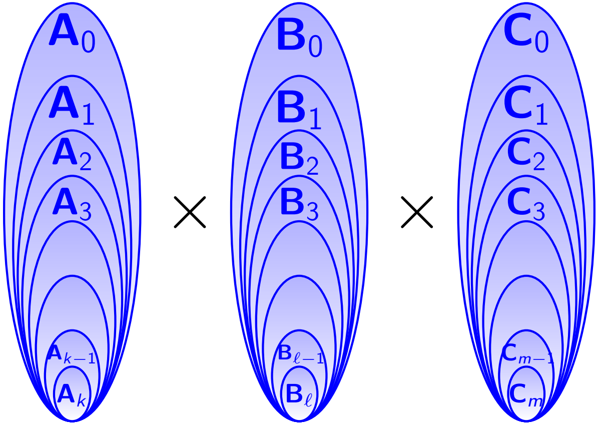
$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



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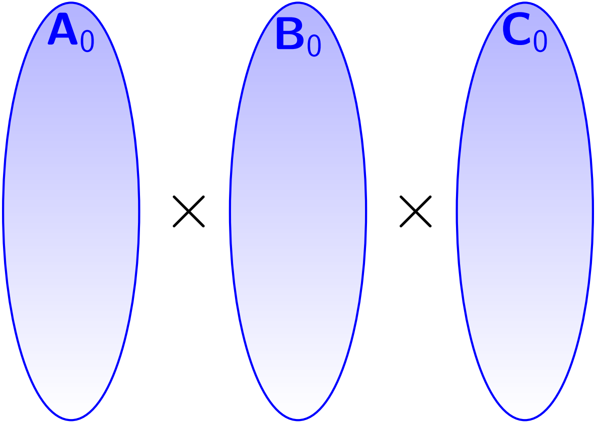

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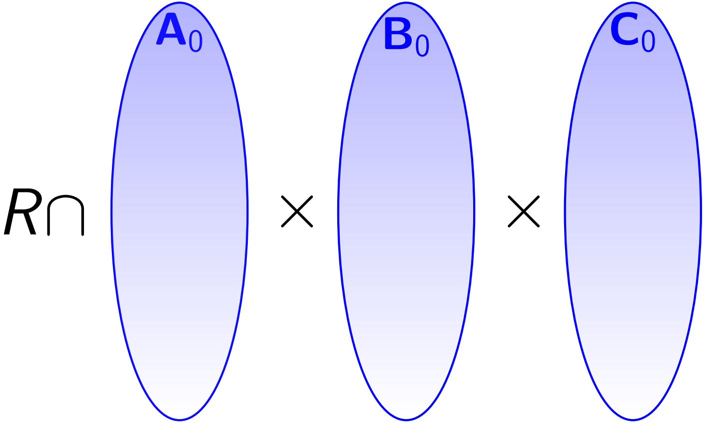

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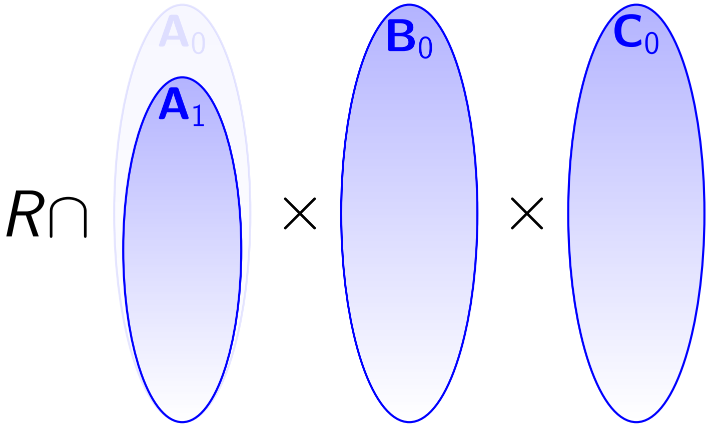

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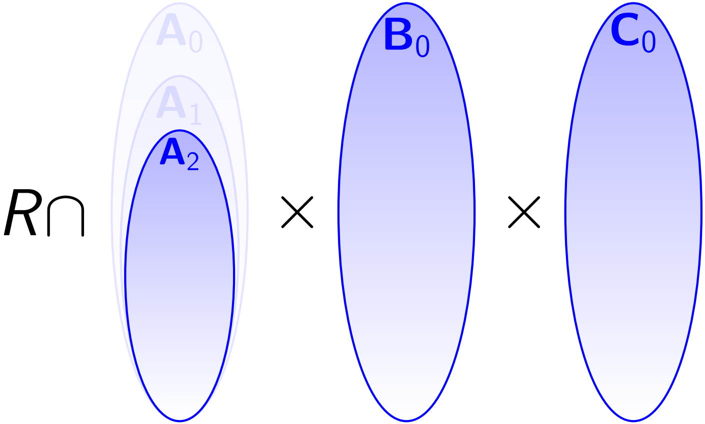

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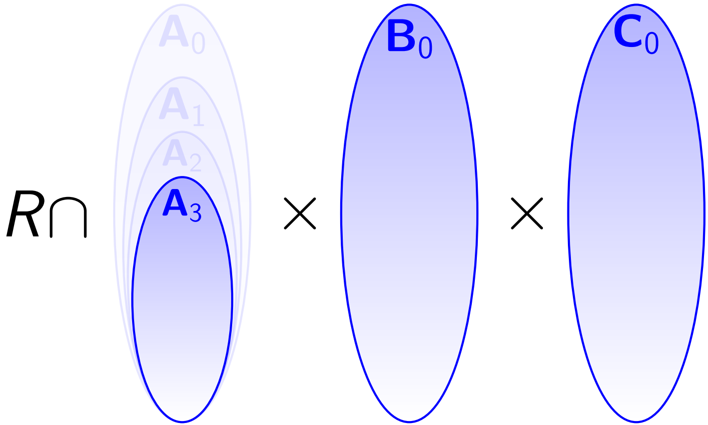

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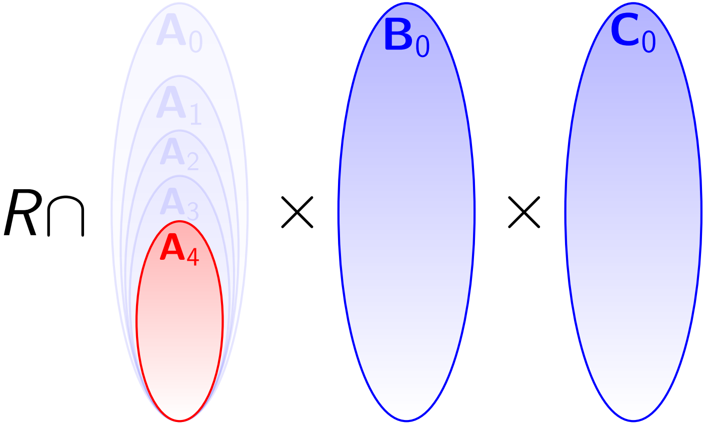

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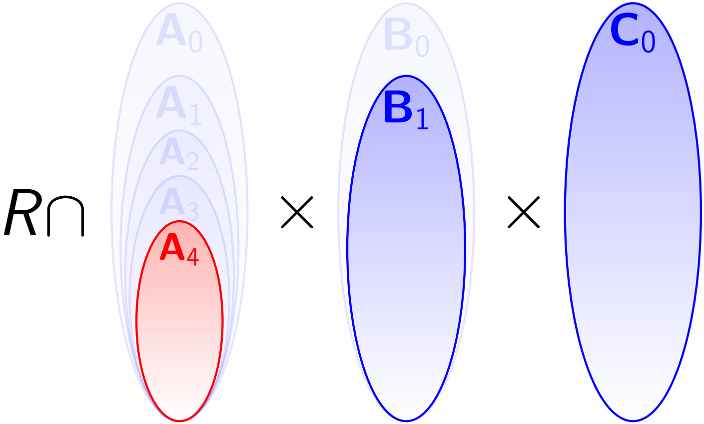

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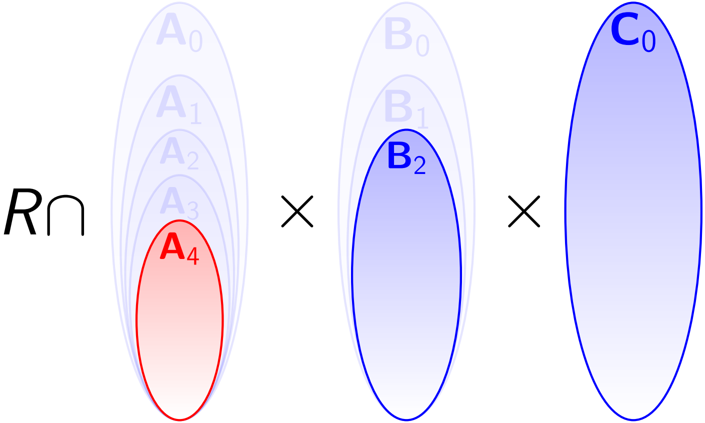

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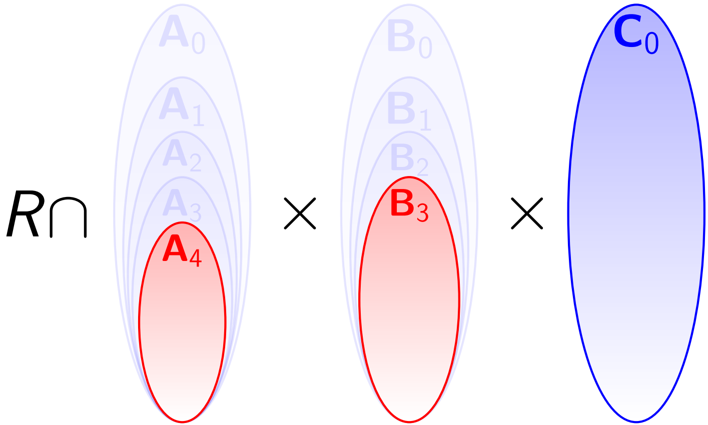

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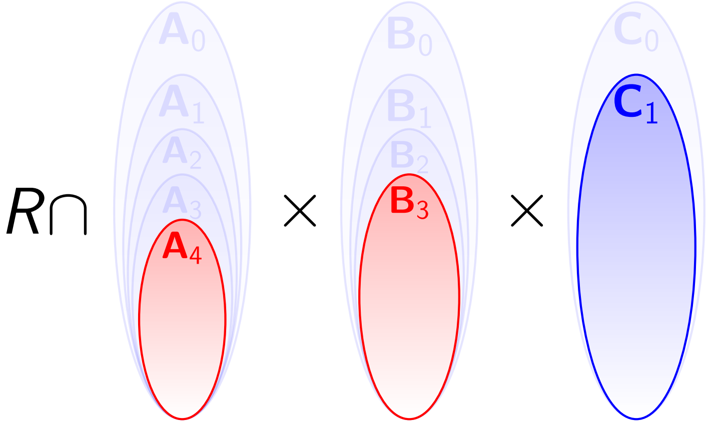

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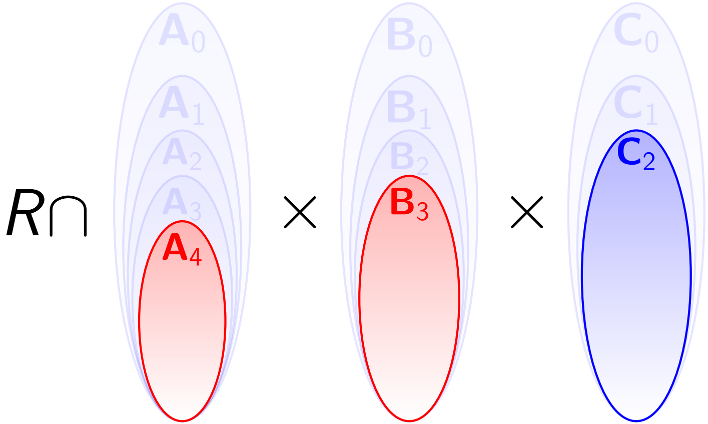

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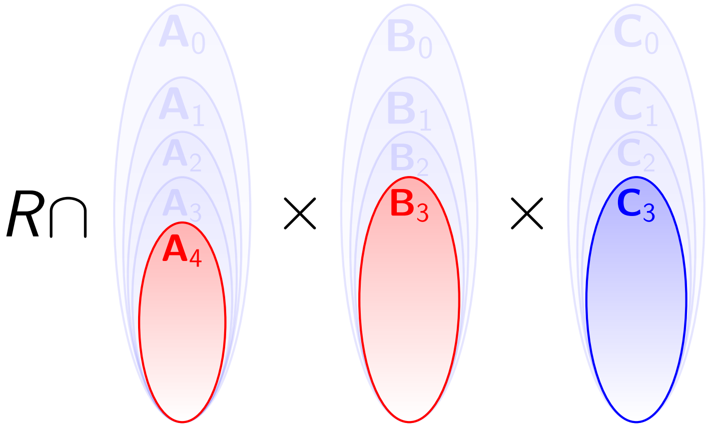

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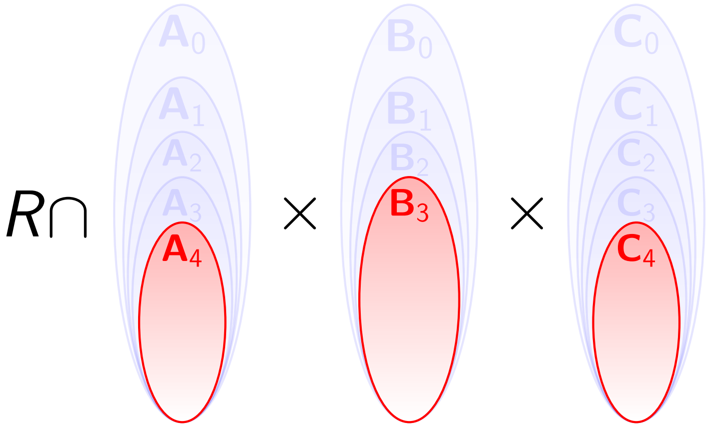

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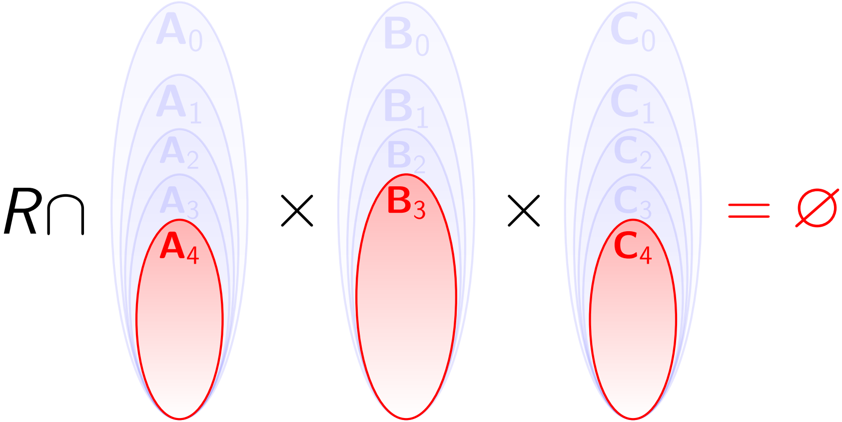

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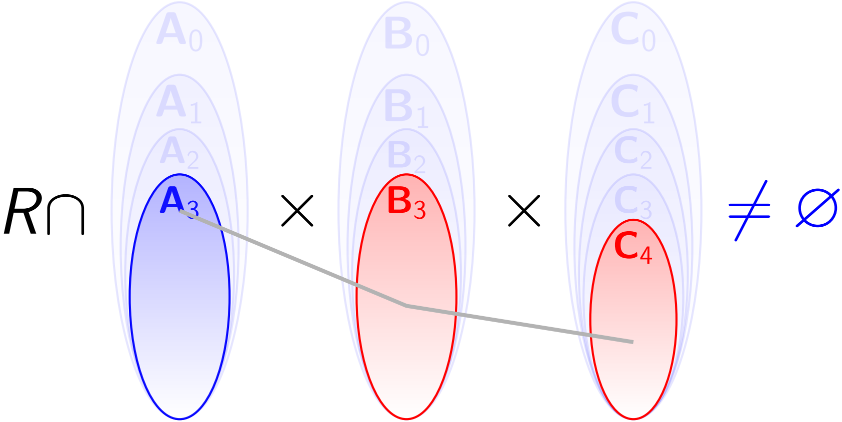

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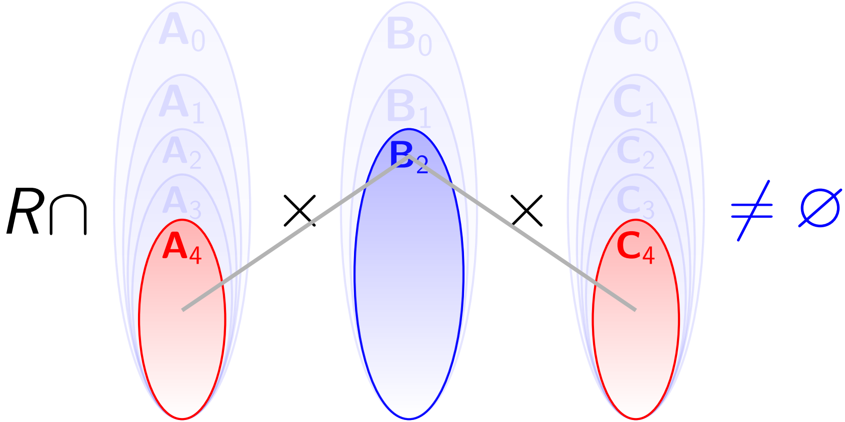

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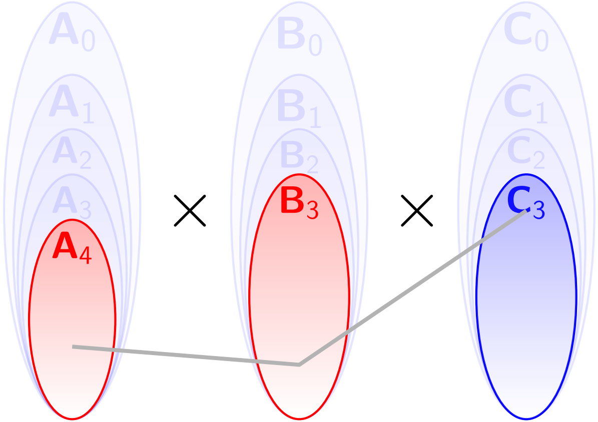

$R \cap \mathbf{A}_4 \times \mathbf{B}_3 \times \mathbf{C}_3 \neq \emptyset$

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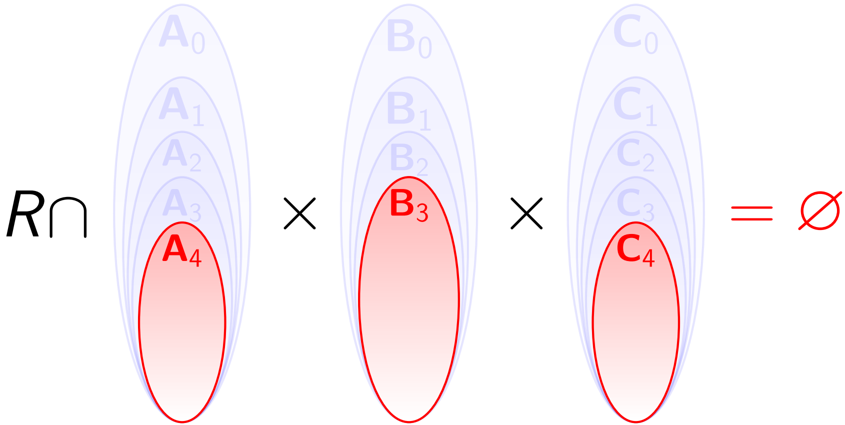

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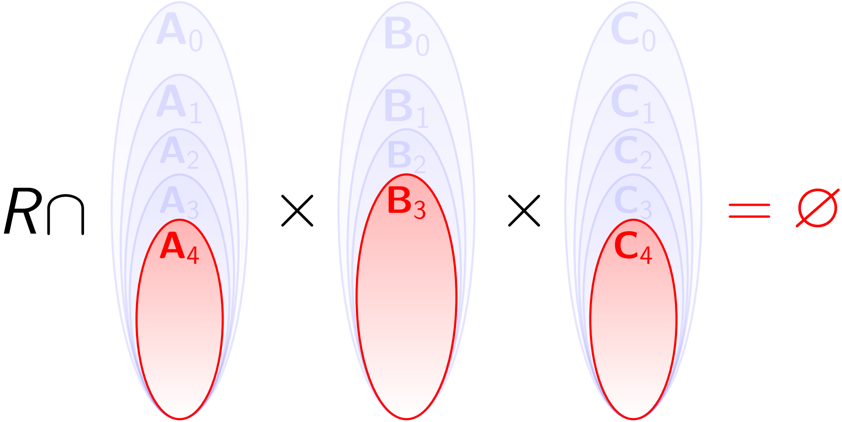

$$\mathbf{A}_4 \blacktriangleleft^{A_0} \mathbf{A}_3$$

Helly property

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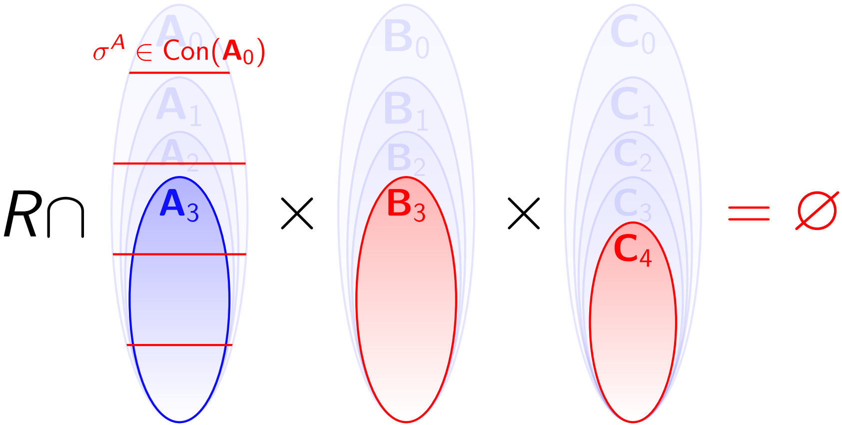

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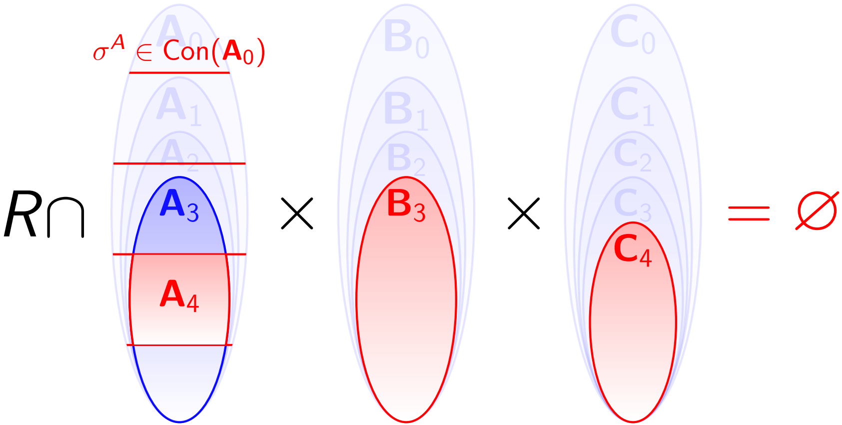

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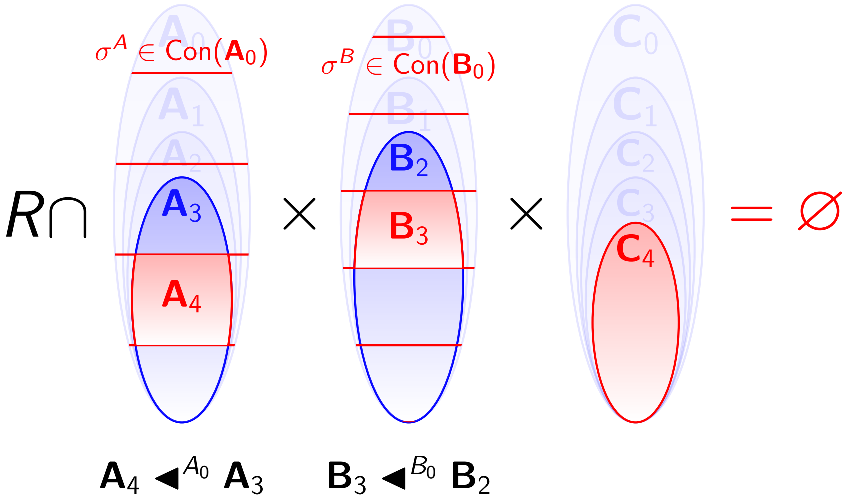

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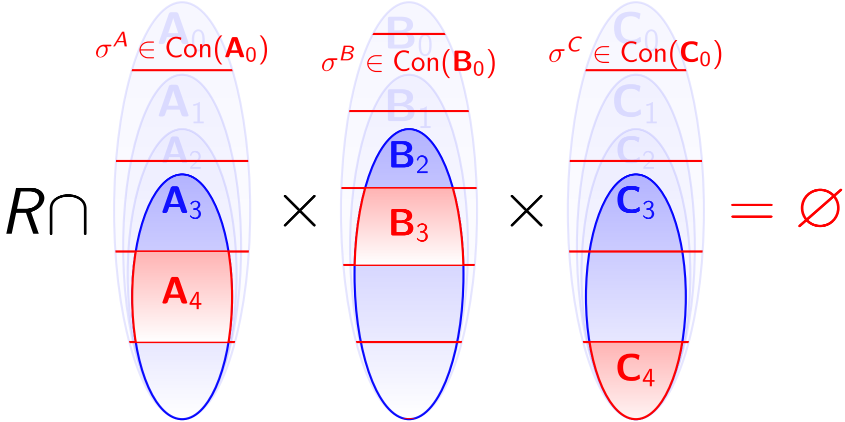

$$\mathbf{A}_4 \blacktriangleleft^{A_0} \mathbf{A}_3 \qquad \mathbf{B}_3 \blacktriangleleft^{B_0} \mathbf{B}_2 \qquad \mathbf{C}_4 \blacktriangleleft^{C_0} \mathbf{C}_3$$

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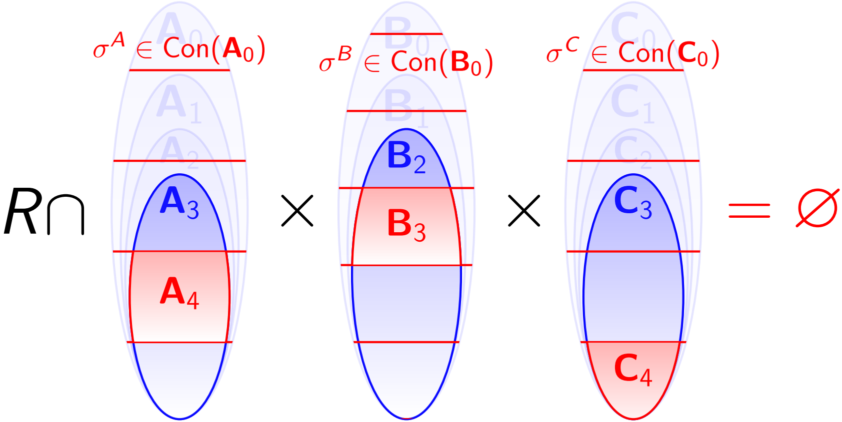

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There exist bridges between congruences σ^A, σ^B, and σ^C

Properties of $\lll^A$

(Transitivity) $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection) $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

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(Helly)

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then

1. either $<_i = \lhd_3$ for every i and $n = 2$,
2. or $<_i = \blacktriangleleft^A$ for every i

Properties of $\lll^A$

(Transitivity) $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection) $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if $f \colon \mathbf{A} \to \mathbf{A}'$ is a surjective homomorphism,

(Pushforward) $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} \mathbf{f}(B)$
(Pullback) $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A \mathbf{f}^{-1}(B')$

(Ubiquity) if $\mathbf{B} \lll^A \mathbf{A}$ and $|\mathbf{B}| > 1$ then $\exists \mathbf{C} \lll^A \mathbf{B}$.

(Helly) Suppose

- $\mathbf{C}_i <_i \mathbf{B}_i \lll^A \mathbf{A}$, $<_i \in \{\prec_2, \lhd_3, \blacktriangleleft^A\}$, $i \in [n]$, $n \geq 2$,
- $\bigcap \mathbf{C}_i = \varnothing$,
- $\forall j \colon \mathbf{B}_j \cap \bigcap_{i \neq j} \mathbf{C}_i \neq \varnothing$.

then

1. either $<_i = \lhd_3$ for every i and $n = 2$,
2. or $<_i = \blacktriangleleft^A$ for every i + exist nice bridges

Bridges

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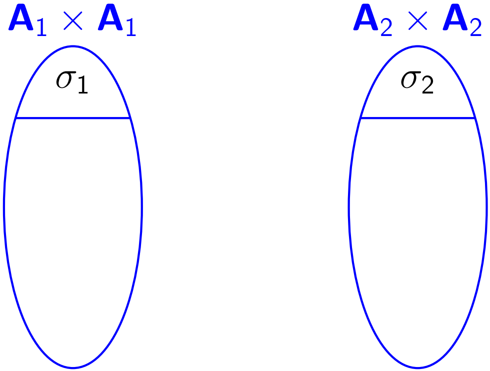

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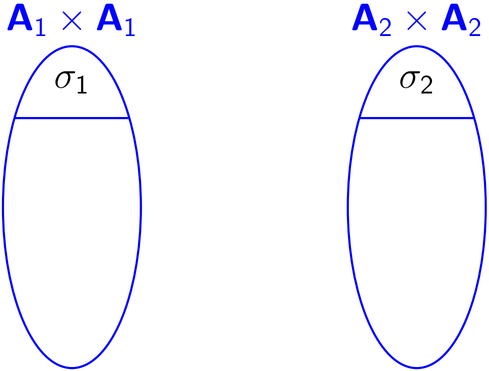

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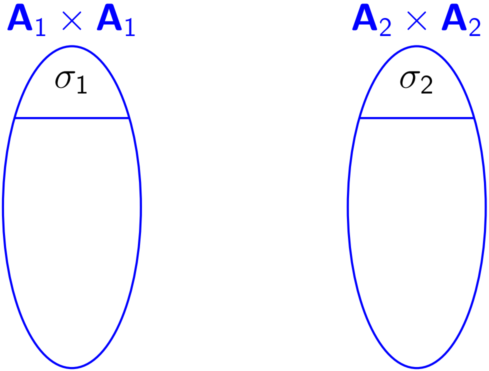
$$\begin{matrix} (a_1, & a_2, & a_3, & a_4) \\ \sigma_1 \wr & \sigma_1 \wr & \wr \sigma_2 & \wr \sigma_2 \\ (b_1, & b_2, & b_3, & b_4) \end{matrix}$$



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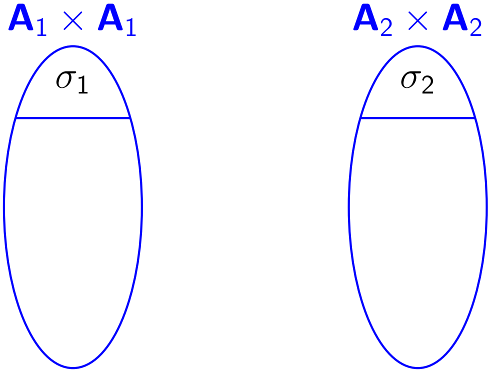

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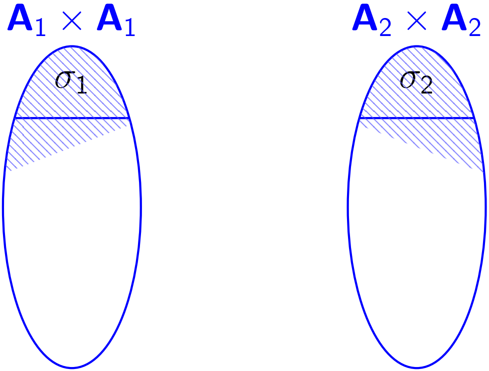

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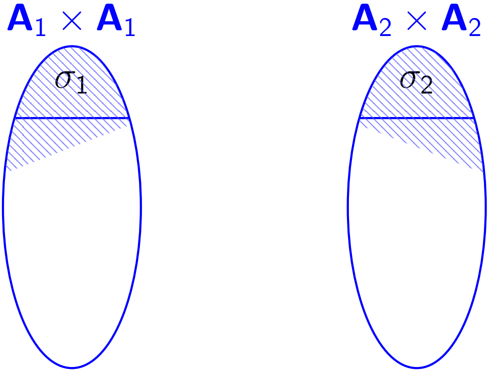

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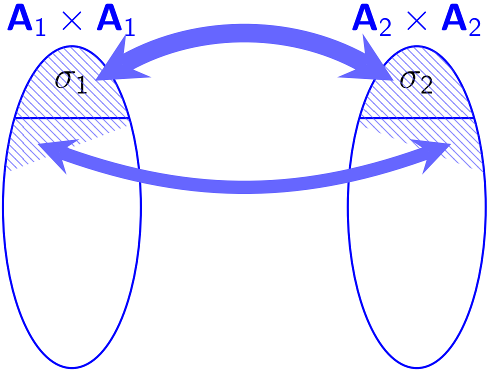

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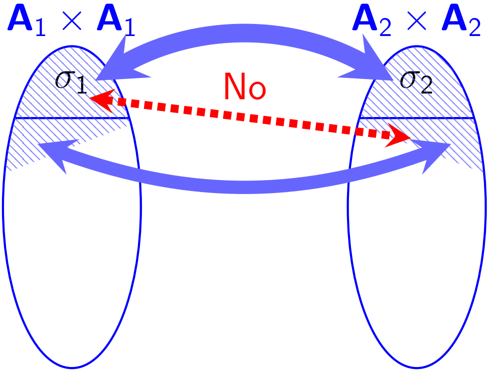

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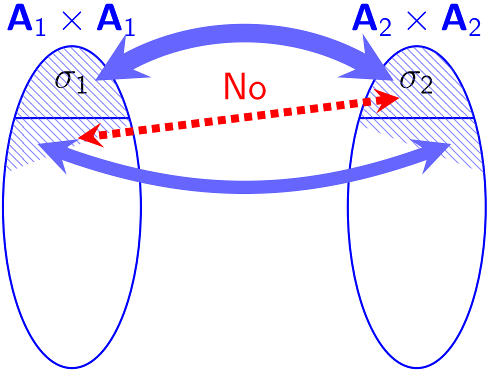

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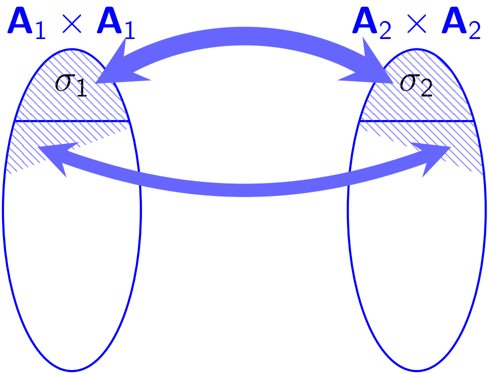

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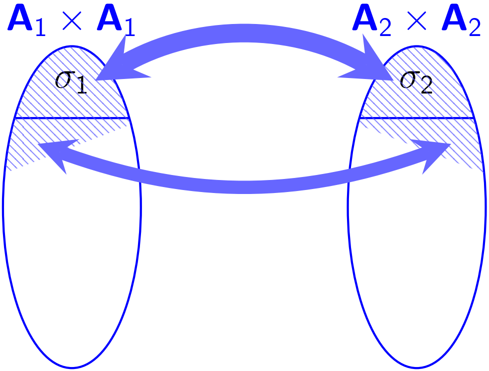

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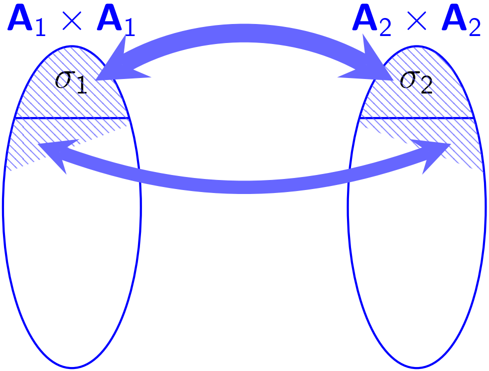

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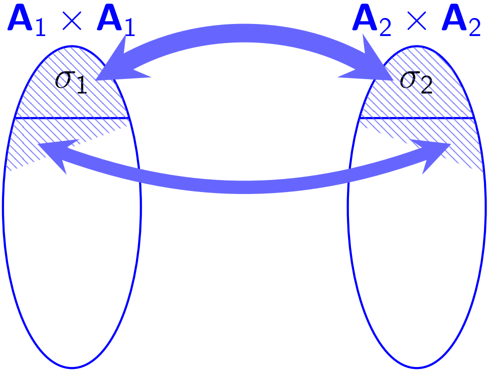

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δ is a bridge between 0-cogruence and $(\mathrm{mod}\ 2)$-congruence.

Composition of bridges

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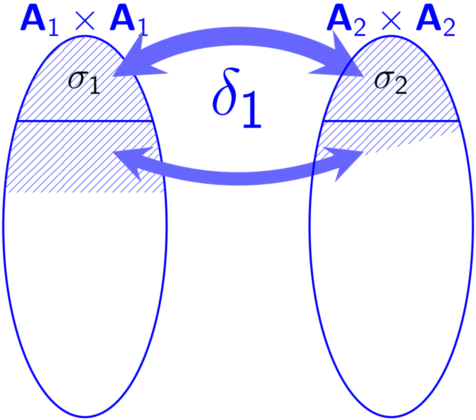

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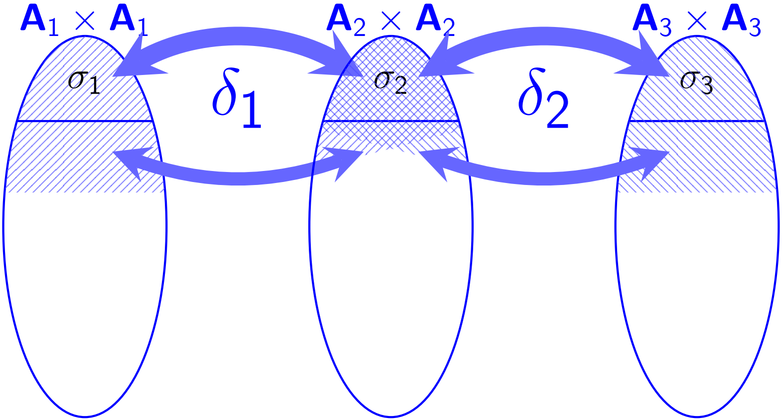
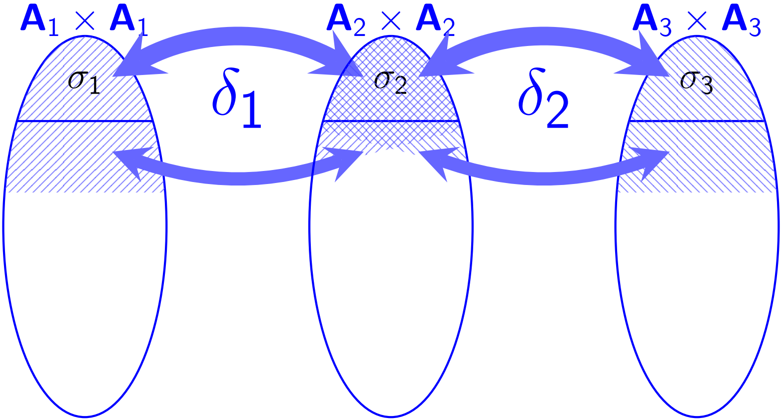


Composition of bridges

δ_1 is a bridge from σ_1 to σ_2
δ_2 is a bridge from σ_2 to σ_3
$\delta := \delta_1 \circ \delta_2$ is a bridge from σ_1 to σ_3 (if σ_2 is irreducible)

$\delta(x_1, x_2, z_1, z_2) := \exists y_1 \exists y_2\ \delta_1(x_1, x_2, y_1, y_2) \wedge \delta_2(y_1, y_2, z_1, z_2)$



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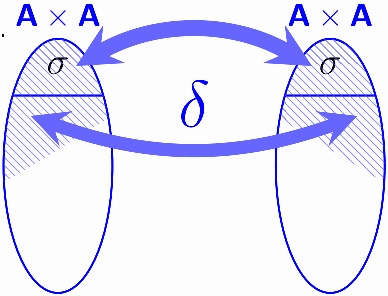

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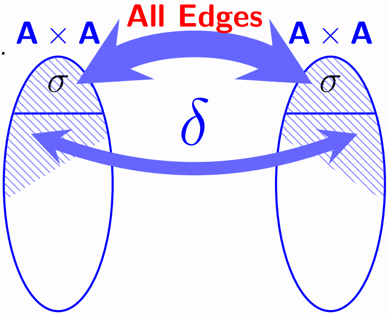

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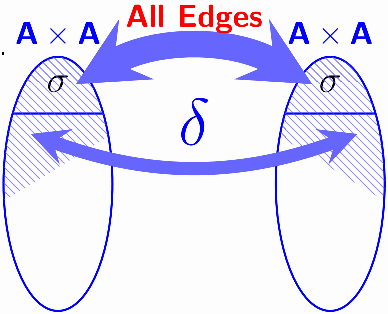

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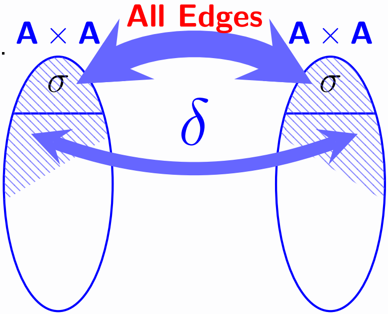

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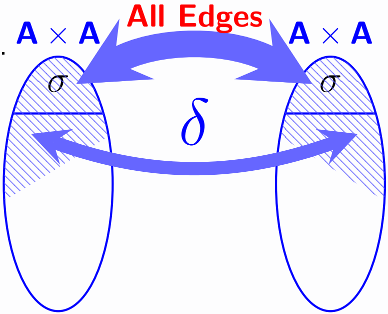

$\mathbf{C} \in \mathbf{B} \boxtimes \mathbb{Z}_p$:

- domain $C = B \times \mathbb{Z}_p$
- $f^{\mathbf{C}}(x_1, \ldots, x_n) = (f^{\mathbf{B}}(x_1^{(1)}, \ldots, x_n^{(1)}),$
  $g_f(x_1^{(1)}, \ldots, x_n^{(1)}) + a_1 x_1^{(2)} + a_2 x_2^{(2)} + \cdots + a_n x_n^{(2)})$

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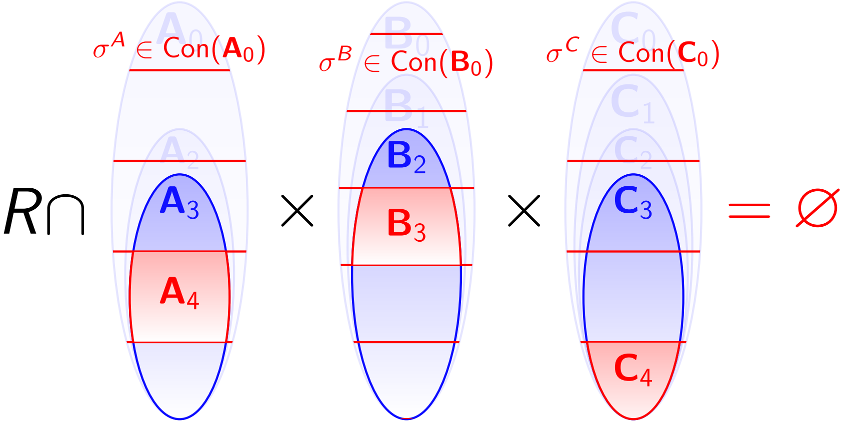
Helly property

$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0$, $\mathbf{A}_k \lll^{A_0} \mathbf{A}_0$, $\mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0$, $\mathbf{C}_m \lll^{C_0} \mathbf{C}_0$.



$\mathbf{A}_4 \blacktriangleleft^{A_0} \mathbf{A}_3$ $\quad$ $\mathbf{B}_3 \blacktriangleleft^{B_0} \mathbf{B}_2$ $\quad$ $\mathbf{C}_4 \blacktriangleleft^{C_0} \mathbf{C}_3$

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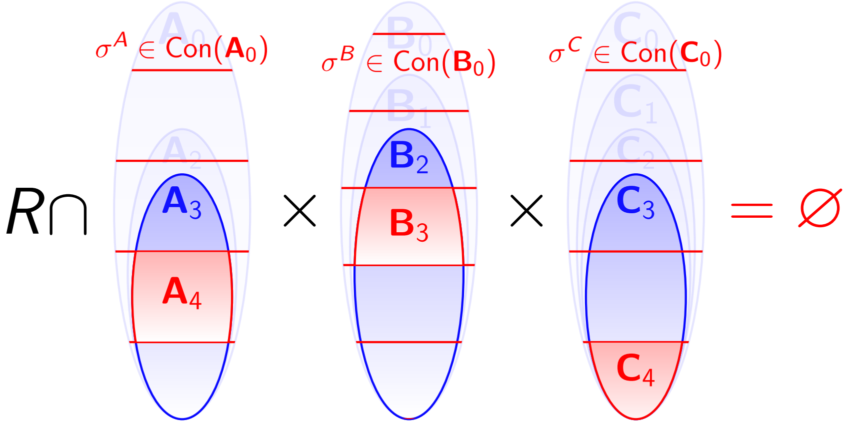

$\mathbf{A}_4 \blacktriangleleft^{A_0} \mathbf{A}_3 \qquad \mathbf{B}_3 \blacktriangleleft^{B_0} \mathbf{B}_2 \qquad \mathbf{C}_4 \blacktriangleleft^{C_0} \mathbf{C}_3$

Theorem

$\mathrm{pr}_{1,2}(R)$ is linked $\Rightarrow \sigma^A$ and σ^B are perfect linear congruences.

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A simplified proof of the CSP Dichotomy Conjecture

12 pages modulo properties of strong subalgebras.

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12 pages modulo properties of strong subalgebras.

New CSP algorithms based on Linear Relaxation

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- Congruences are global!
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Thank you for your attention