Three Fundamental Questions in Modern Infinite-Domain Constraint Satisfaction

Michael Pinsker, Jakub Rydval, Moritz Schöbi, Christoph Spiess

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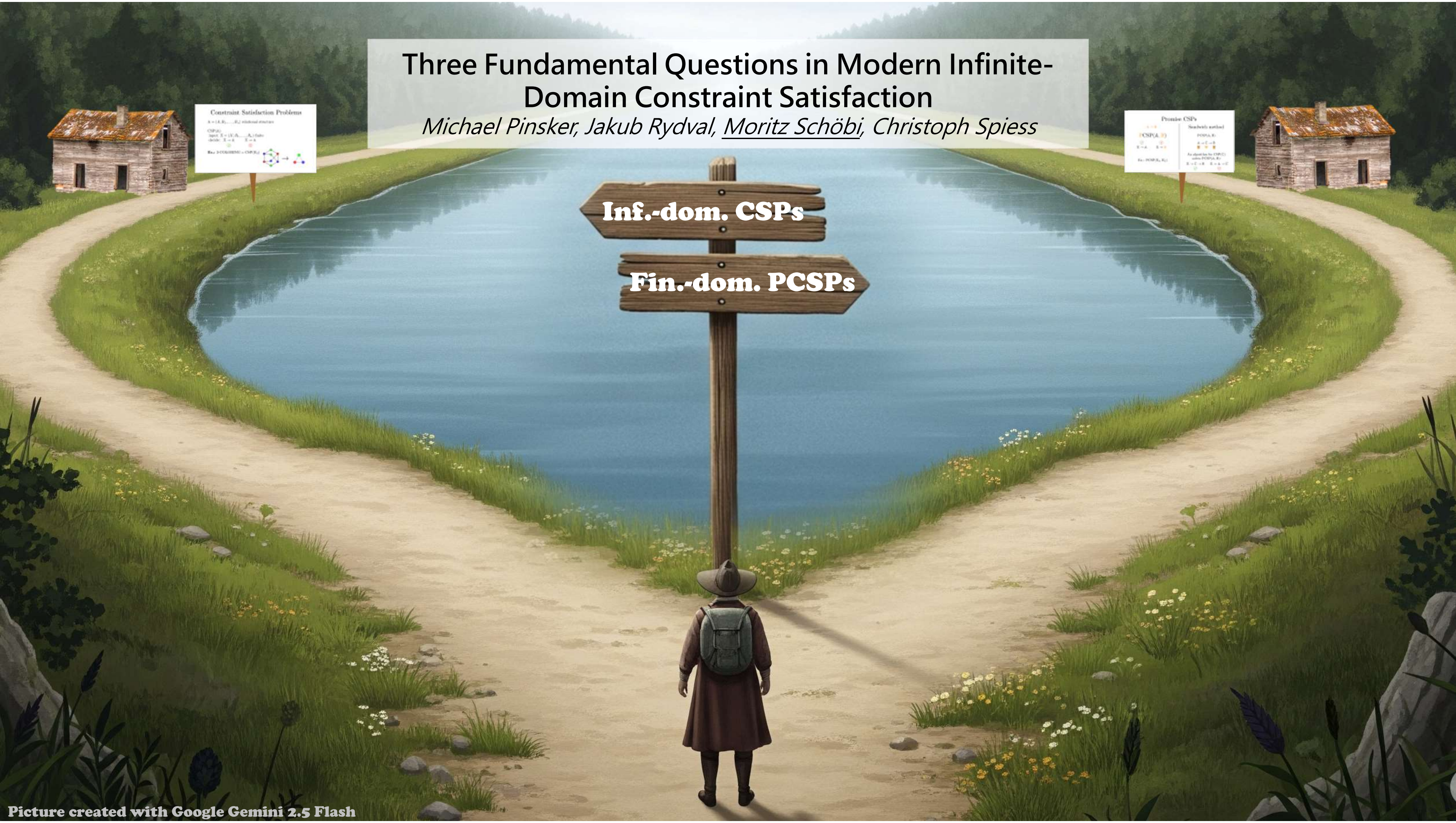

Constraint Satisfaction Problems

$\mathbb{A} = (A; R_1, \ldots, R_n)$ relational structure

CSP($\mathbb{A}$)

input: $\mathbb{X} = (X; R_1, \ldots, R_n)$ finite

decide: $\mathbb{X} \to \mathbb{A}$ ✓ $\quad \mathbb{X} \not\to \mathbb{A}$ ✗

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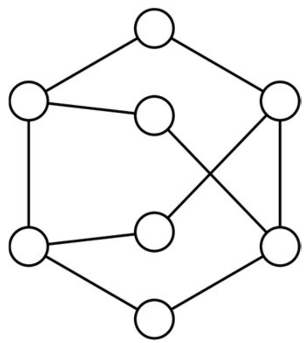
Ex.: 3-COLORING



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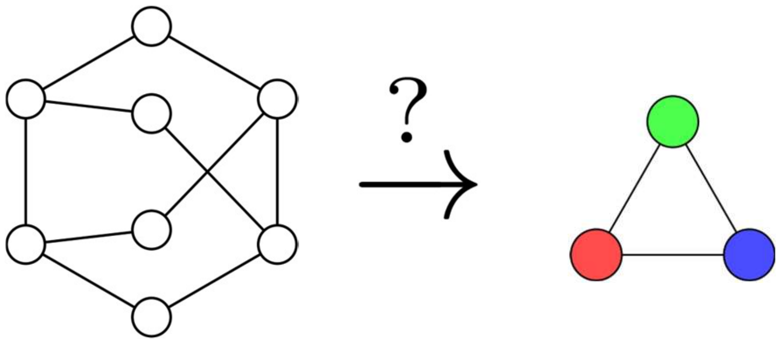

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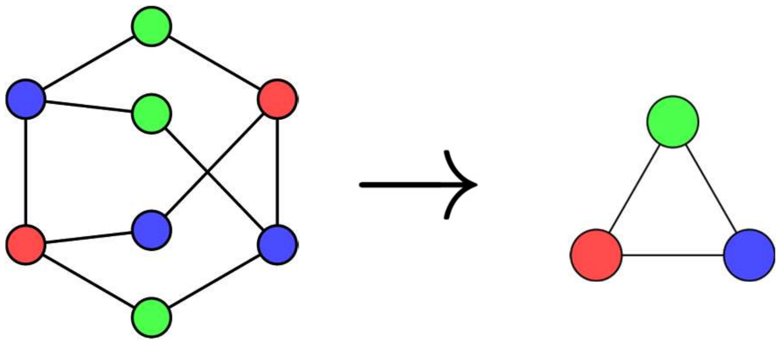

Constraint Satisfaction Problems

$\mathbb{A} = (A; R_1, \ldots, R_n)$ relational structure

$\mathbb{A}^n \ldots$ *n-th power* of $\mathbb{A}$

$\mathrm{Pol}(\mathbb{A}) \coloneqq \bigcup_{n \in \mathbb{N}} \mathrm{Hom}(\mathbb{A}^n, \mathbb{A})$

$\mathrm{Pol}(\mathbb{A}) \models s(x_1, \ldots, x_n) \approx t(y_1, \ldots, y_m) :\Leftrightarrow \exists s^{\mathbb{A}}, t^{\mathbb{A}} \in \mathrm{Pol}(\mathbb{A})$
$\forall a_1, \ldots, a_n, b_1, \ldots, b_m \in A \colon s^{\mathbb{A}}(a_1, \ldots, a_n) = t^{\mathbb{A}}(b_1, \ldots, b_m)$

Constraint Satisfaction Problems

$\mathbb{A} = (A; R_1, \ldots, R_n)$ relational structure

$\mathbb{A}^n \ldots$ *n-th power* of $\mathbb{A}$

pp formulas$\ldots \exists x_1 \ldots \exists x_n \bigwedge_i \phi_i(x_1, \ldots, x_n)$

$\mathbb{A}$ *pp-constructs* $\mathbb{B} \Leftrightarrow \exists \mathbb{B}'$: $\mathbb{A}$ pp-defines $\mathbb{B}'$ on $A^n \wedge \mathbb{B}' \sim_H \mathbb{B}$.

Fact: If $\mathbb{A}$ pp-constructs $\mathbb{B}$, and CSP($\mathbb{A}$) is in P, so is CSP($\mathbb{B}$).

Finite Dichotomy Theorem

Theorem (Bulatov '17 and Zhuk '17) $\mathbb{A}$ *finite relational structure. Then*

- $\mathbb{A}$ *pp-constructs* $\mathbb{K}_3$ *and* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete, or*
- $\mathrm{Pol}(\mathbb{A}) \models s(x, y, z, x, y, z) \approx s(y, z, x, z, x, y)$ *and* $\mathrm{CSP}(\mathbb{A})$ *is tractable.*

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Extend result to countable structures?

$\mathbb{A}$ is *homogeneous* if isomorphisms between finite substructures extend to global automorphisms.
$\mathbb{A}$ is *finitely bounded* if class of finite substructures is given by uniform universal f.o. axiomatisation.

Ex.: $(\mathbb{Q};<)$ is fbh:

- Order-preserving functions extend to automorphisms.
- Axioms irreflexivity, transitivity and totality.

An infinite analogon?

Conjecture (Bodirsky and Pinsker '12) $\mathbb{A}$ *reduct of countable fbh* $\mathbb{B}$*. Exactly one holds.*

- $\mathbb{A}$ *pp-constructs* $\mathbb{K}_3$ *(*$\Rightarrow \mathrm{CSP}(\mathbb{A})$ *is NP-complete);*
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Note: $s(x,y,z,x,y,z) \approx s(y,z,x,z,x,y)$ changed to $\alpha \circ s(x,y,z,x,y,z) \approx \beta \circ s(y,z,x,z,x,y)$.

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Impose additional meaningful structural and algebraic assumptions on $\mathbb{A}$ and $\mathrm{Pol}(\mathbb{A})$ w.l.o.g.?

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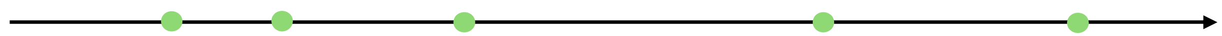

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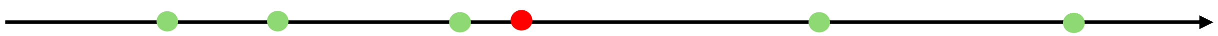

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Removing Algebraicity

Theorem *$\mathbb{A}$ reduct of fbh $\mathbb{B}$. There is $\mathbb{A}'$, reduct of fbh $\mathbb{B}'$ such that:*

- *$\mathbb{A}', \mathbb{B}'$ have no algebraicity.*
- *$\mathrm{CSP}(\mathbb{A}')$ and $\mathrm{CSP}(\mathbb{A})$ are Datalog-interreducible.*
- *$\mathbb{A}'$ pp-constructs $\mathbb{K}_3$ iff $\mathbb{A}$ does.*
- *$\mathrm{Pol}(\mathbb{A}')$ preserves $\neq$.*

$\mathbb{A}$ red. of $\mathbb{B}$

$\updownarrow$ Datalog

$\mathbb{A}'$ red. of $\mathbb{B}'$

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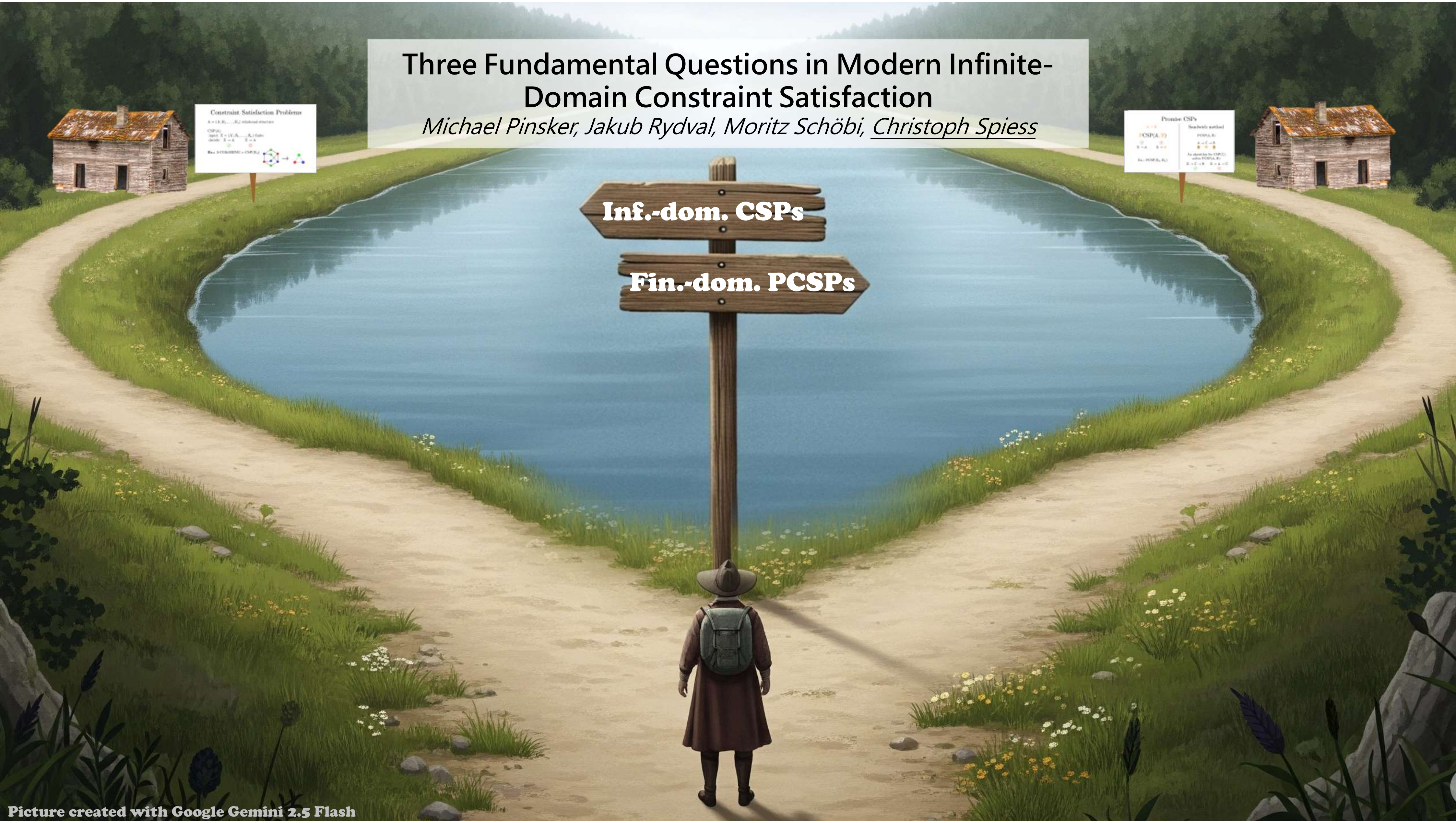

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Promise CSPs

Promise CSPs

$\text{CSP}(\mathbb{A})$

✓ $\mathbb{X} \to \mathbb{A}$

✗ $\mathbb{X} \not\to \mathbb{A}$

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

$$\mathrm{CSP}(\mathbb{A})$$

✓ $\mathbb{X} \to \mathbb{A}$

✗ $\mathbb{X} \not\to \mathbb{A}$

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

$$\text{PCSP}(\mathbb{A}, \mathbb{B})$$

✓ $\mathbb{X} \to \mathbb{A}$

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Ex.: $\text{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

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Sandwich method

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

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✓ $\mathbb{X} \to \mathbb{A}$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

Promise CSPs

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Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

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Promise CSPs

$\mathbb{A} \to \mathbb{B}$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

An algorithm for $\mathrm{CSP}(\mathbb{C})$ solves $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$:

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

$$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$$

✓ $\mathbb{X} \to \mathbb{A}$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

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$\mathbb{X} \to \mathbb{C}$ $\qquad$ $\mathbb{X} \not\to \mathbb{C}$

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

✓ $\mathbb{X} \to \mathbb{A}$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

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$\mathbb{X} \to \mathbb{C} \to \mathbb{B}$ $\quad$ $\mathbb{X} \not\to \mathbb{C}$

Promise CSPs

$\mathbb{A} \to \mathbb{B}$

$$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

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Promise CSPs

$\mathbb{A} \to \mathbb{B}$

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

✓ $\mathbb{X} \to \mathbb{A}$

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Ex.: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

An algorithm for $\mathrm{CSP}(\mathbb{C})$ solves $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$:

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Sandwich method

$\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

$\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

An algorithm for $\mathrm{CSP}(\mathbb{C})$ solves $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$:

$\mathbb{X} \to \mathbb{C} \to \mathbb{B}$ ✓

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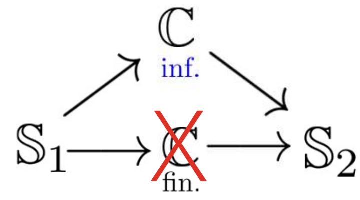
Infinite cheeses

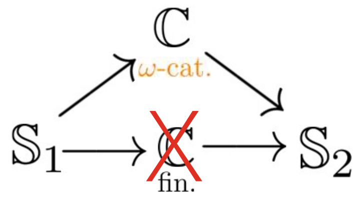
Infinite cheeses

Theorem (Barto '19, Mottet '25)
$\exists$ *finite* $(\mathbb{S}_1, \mathbb{S}_2)$

- $\exists$ *infinite*/ω-*categorical tractable cheese* $\mathbb{C}$
- $\nexists$ *finite tractable cheese*



Infinite cheeses



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$\mathbb{A}$ ω-categorical
$|\{\mathcal{O}(\bar{a}) : \bar{a} \in A^k\}| < \infty$
$\mathcal{O}(\bar{a}) \coloneqq \{\alpha(\bar{a}) : \alpha \in \mathrm{Aut}(\mathbb{A})\}$

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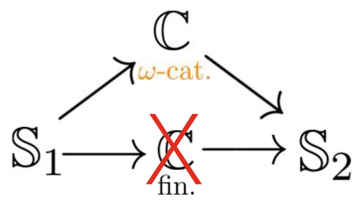
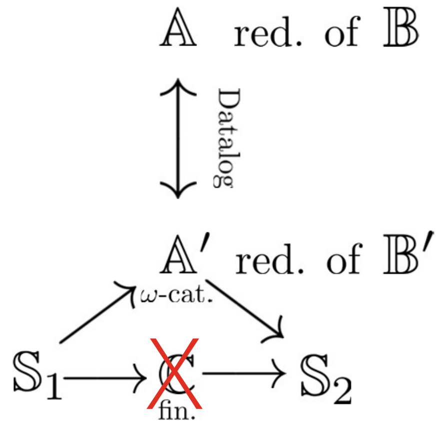


Theorem
$\mathbb{A}$ *reduct of fbh* $\mathbb{B}$
$\Rightarrow \exists \mathbb{A}'$ *reduct of fbh* $\mathbb{B}'$, *finite* $\mathbb{S}_1, \mathbb{S}_2$

- $\mathrm{CSP}(\mathbb{A})$ *and* $\mathrm{CSP}(\mathbb{A}')$ *Datalog-interreducible.*
- $\mathbb{A}'$ *is a cheese for* $\mathrm{PCSP}(\mathbb{S}_1, \mathbb{S}_2)$
- $\nexists$ *finite tractable cheese*



Sandwich Recipe

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$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, $\mathrm{Pol}(\mathbb{A})$ preserves $\neq$

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Sandwich Recipe

$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, Pol($\mathbb{A}$) preserves $\neq$

- $\mathbb{A} \to \mathbb{A}^{[d]}$

Sandwich Recipe

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$\mathbb{A} \to \mathbb{A}^{[d]}$ $\Bigg\{$

Sandwich Recipe

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$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, $\mathrm{Pol}(\mathbb{A})$ preserves $\neq$

- $\mathbb{S}^{[d]}$ for fin. $\mathbb{S} \leq \mathbb{A}$
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$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, $\mathrm{Pol}(\mathbb{A})$ preserves $\neq$

$\mathbb{S}^{[d]}$ for fin. $\mathbb{S} \leq \mathbb{A}$

$\mathbb{A} \to \mathbb{A}^{[d]}$

- $\mathrm{Pol}(\mathbb{A}^{[d]}) = \mathrm{Pol}(\mathbb{A}) \curvearrowright A^d$
- CSP($\mathbb{A}$) and CSP($\mathbb{A}^{[d]}$) are "the same" (up to pp-interpretation)
- If $\mathbb{A}$ is (ω-categorical, homogeneous, finitely bounded, a reduct of $\mathbb{B}$), then $\mathbb{A}^{[d]}$ is (ω-categorical, homogeneous, finitely bounded, a reduct of $\mathbb{B}^{[d]}$)

Sandwich Recipe

$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, Pol($\mathbb{A}$) preserves $\neq$

- $\mathbb{S}^{[d]}$ for fin. $\mathbb{S} \leq \mathbb{A}$
- $\mathbb{A} \to \mathbb{A}^{[d]}$
- $\mathbb{A}^{[d]}_{/\mathrm{Aut}(\mathbb{B})}$

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- $\mathbb{A}^{[d]}_{/\mathrm{Aut}(\mathbb{B})}$

$$(\mathbb{Q};<)^{[3]}_{/\mathrm{Aut}((\mathbb{Q};<))} = \left\{ \left[\begin{pmatrix}0\\0\\0\end{pmatrix}\right], \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right], \left[\begin{pmatrix}2\\1\\0\end{pmatrix}\right], \left[\begin{pmatrix}1\\1\\0\end{pmatrix}\right], \left[\begin{pmatrix}0\\1\\0\end{pmatrix}\right], \dots \right\}$$

Sandwich Recipe

$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, $\mathrm{Pol}(\mathbb{A})$ preserves $\neq$

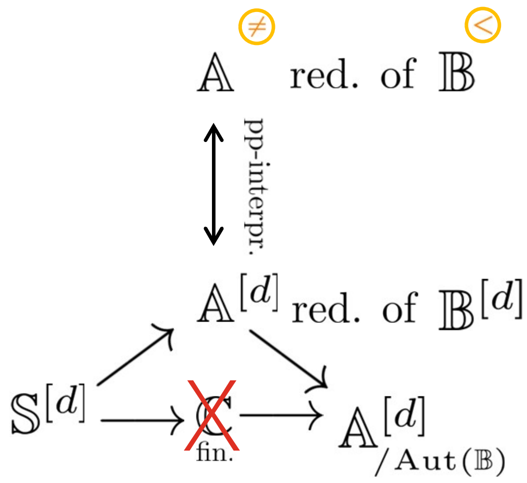
$\mathbb{S}^{[d]}$ for fin. $\mathbb{S} \leq \mathbb{A}$

$\mathbb{A} \longrightarrow \mathbb{A}^{[d]}$

$\mathbb{A}^{[d]}_{/\mathrm{Aut}(\mathbb{B})}$

Proposition
$\mathbb{A}$ reduct of linearly ordered fbh $\mathbb{B}$, $\mathrm{Pol}(\mathbb{A})$ preserves $\neq$.
For large enough $d \in \mathbb{N}$ and $\mathbb{S} \leq \mathbb{A}$ with $|S| \geq 3$:

- *$\mathbb{A}^{[d]}$ is a cheese for $\mathrm{PCSP}(\mathbb{S}^{[d]}, \mathbb{A}^{[d]}_{/\mathrm{Aut}(\mathbb{B})})$;*
- *$\nexists$ finite tractable cheese*



Mise en Place

Proposition
$\mathbb{A}''$ reduct of linearly ordered fbh $\mathbb{B}''$, $\mathrm{Pol}(\mathbb{A}'')$ preserves $\neq$.
For large enough $d \in \mathbb{N}$ and $\mathbb{S}'' \leq \mathbb{A}''$ with $|S''| \geq 3$:

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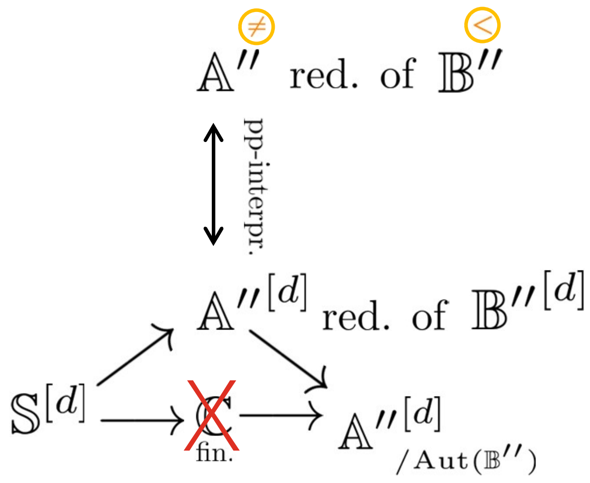
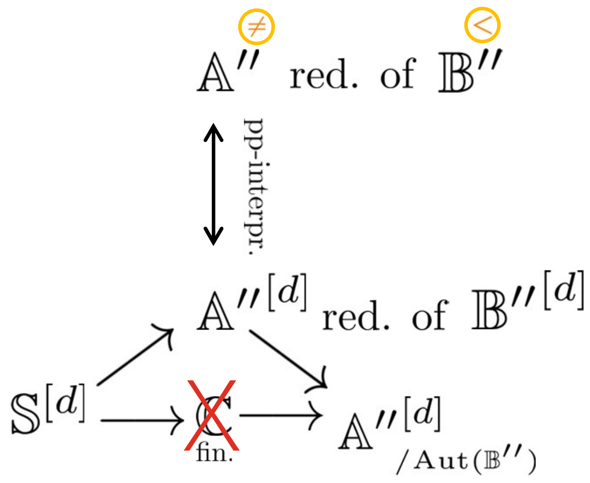


Mise en Place

$\mathbb{A}$ red. of $\mathbb{B}$



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Theorem (Removing algebraicity) *From the first part.*

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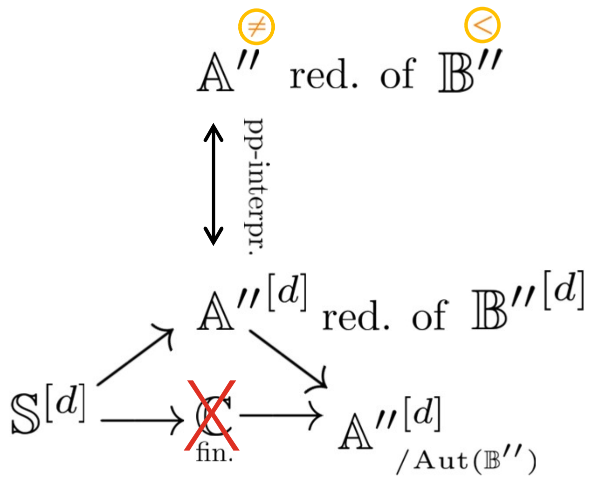

Mise en Place

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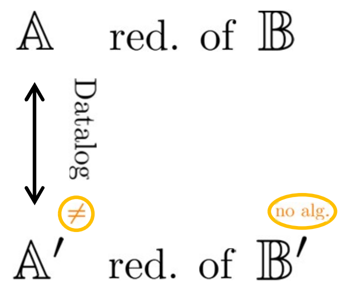

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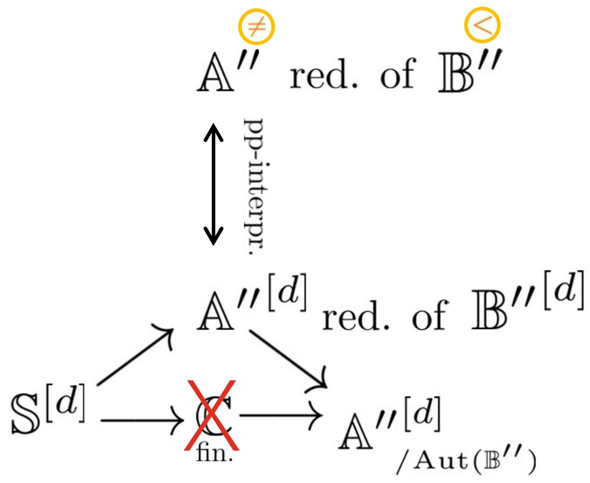

Mise en Place

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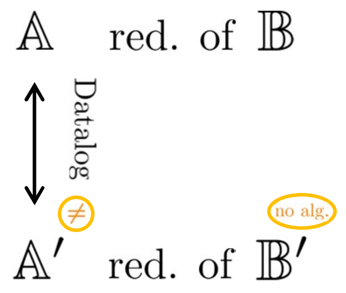
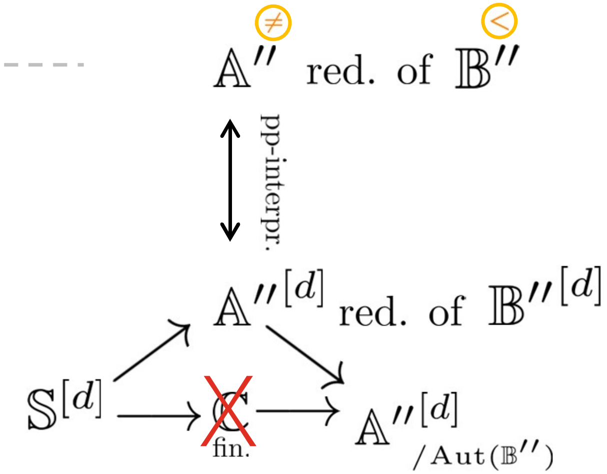


Generic superposition $\mathbb{B}' * (\mathbb{Q}; <)$**:** Possible since $\mathbb{B}'$ has no algebraicity.



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Mise en Place

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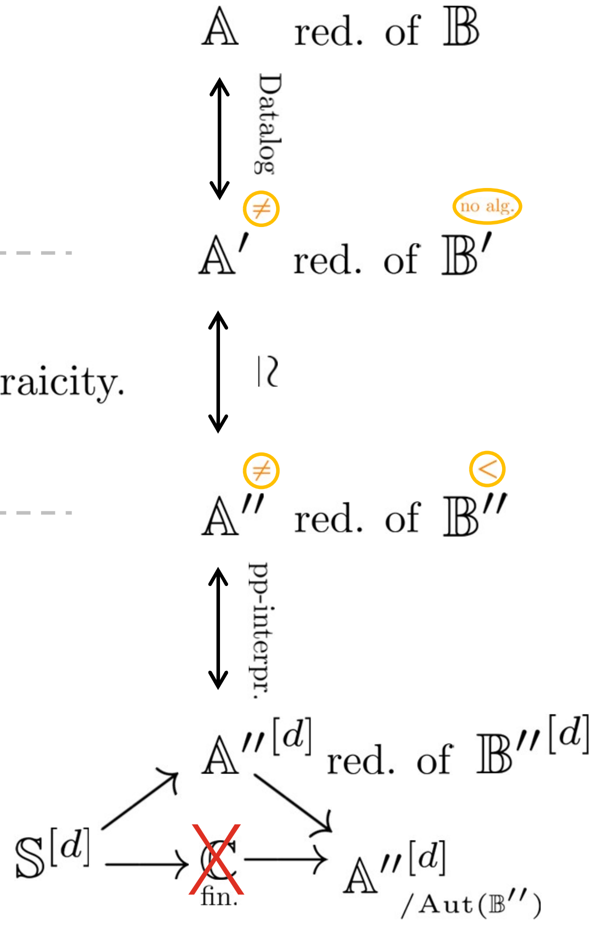
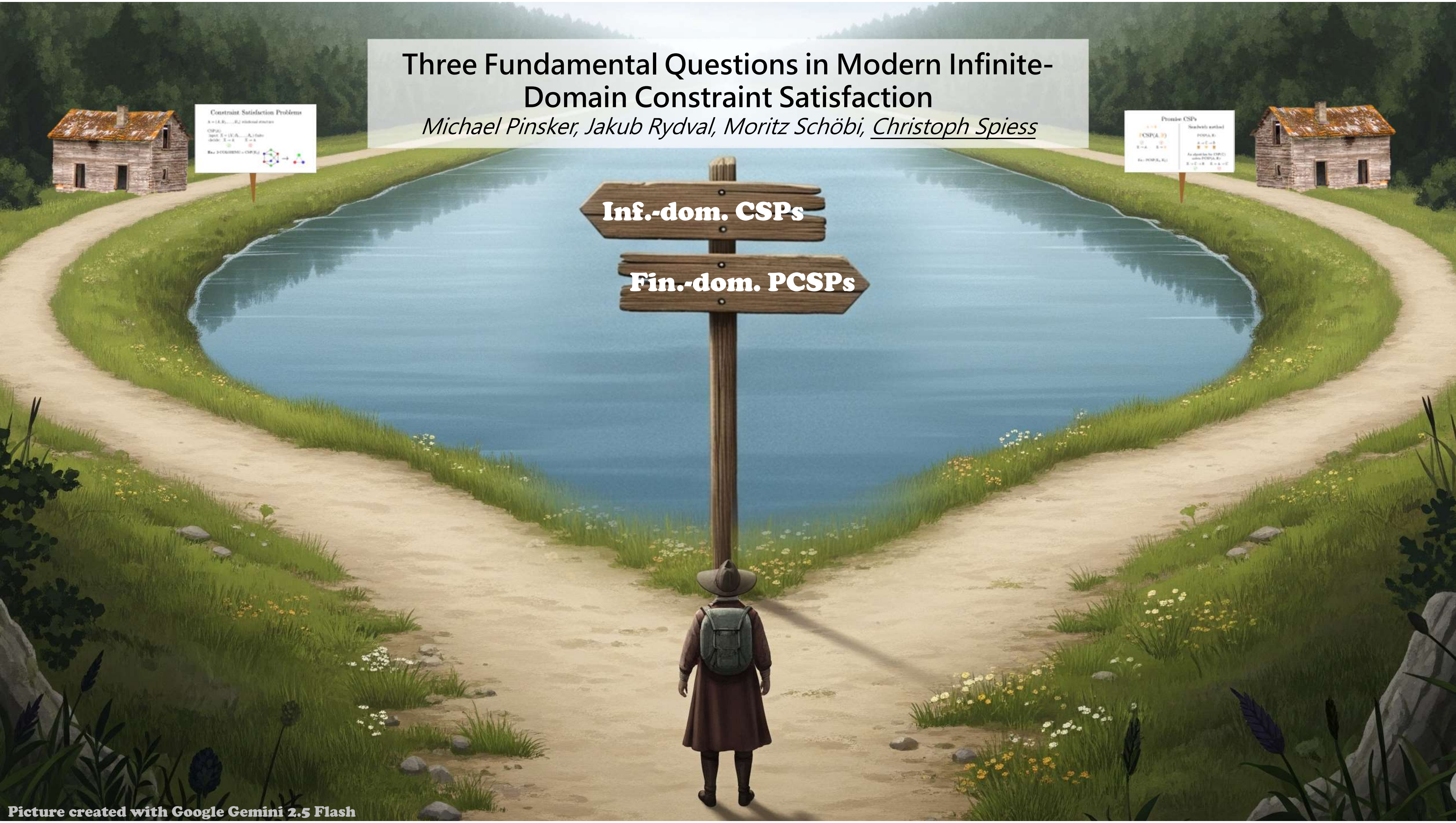


Three Fundamental Questions in Modern Infinite-Domain Constraint Satisfaction

Michael Pinsker, Jakub Rydval, Moritz Schöbi, <u>Christoph Spiess</u>



Picture created with Google Gemini 2.5 Flash


Constraint Satisfaction Problems
Promise CSPs
Sandwich method
Inf.-dom. CSPs
Fin.-dom. PCSPs
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